Using A Hands-On Activities Based Approach in Geometry

Exeter Mathematics Institute

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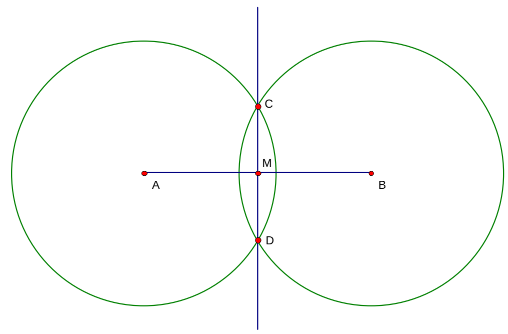
Compass and Straight edge Constructions

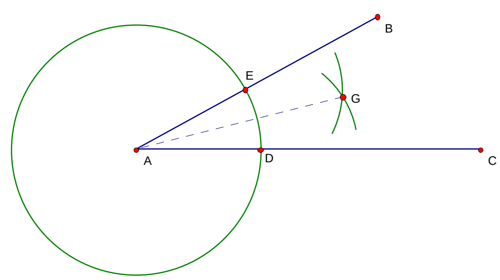
Materials needed are compass and a straight edge.

Comment: If these are done early in a course, then a formal proof of why they work is not appropriate. However, after triangle congruence has been covered, it is worthwhile to revisit these constructions and to nail down why they are guaranteed to work.



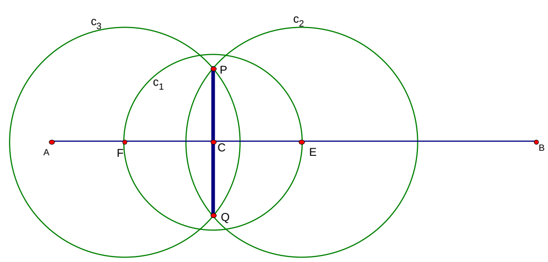
I. Midpoint and perpendicular bisector construction

1. Draw a line segment *AB* on a piece of paper.
2. Use a compass to draw two circles of equal radii that are greater than half the length of the segment *AB* and centered at *A* and *B* respectively. *(Although the picture at right shows full circles, using just the appropriate arcs is sufficient and will make the diagram less crowded.)*
3. Draw a line that connects the two intersection points *C* and *D* of the circles.
4. The line *CD* is the perpendicular bisector of *AB*, and thus intersects *AB* at its midpoint *M*.
5. Try and prove or explain in the space below why this construction produces the midpoint and perpendicular bisector.



II. Angle bisector construction.

1. Draw line segments *AB* and *AC* on a piece of paper as shown at right.
2. Use a compass to draw a circle centered at *A* and intersecting *AB* at *E* and *AC* at *D*.
3. Draw two equal radii circular arcs centered at *E* and *D* and intersecting at point *G*.
4. The ray *AG* is the angle bisector of angle *BAC*
5. Try and prove or explain why this construction produces the angle bisector.



III. Constructing a perpendicular line to a line segment AB through a given point C on the segment.

1. Draw a circle centered at C that intersects the line at E and F.

2. Draw circles of equal radii centered at E and F. The radii must be greater than half the distance from E to F.

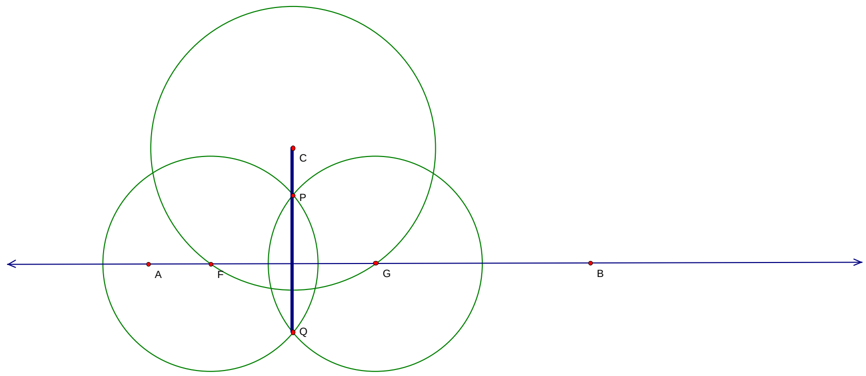
3. The two circles drawn in step #2 intersect at points P and Q. Points P, Q and C are collinear, with the line containing them perpendicular to the original line segment AB.

4. Try and prove or explain why this construction produces a perpendicular line.

IV. Constructing a perpendicular line to a given line *AB* through a point *C* that is NOT on *AB*.
(Although the picture below shows full circles, using just the appropriate arcs is sufficient and will make the diagram less crowded.)

1. Draw a circle centered at *C* with a radius large enough so that it intersects the line *AB* at two points *F* and *G*.

2. Draw circles of equal radii centered at F and G. The radii must be greater than half the distance from F to G.



3. The two circles drawn in step #2 intersect at points P and Q. Points P, Q and C are collinear, with the line containing them perpendicular to the original line AB.

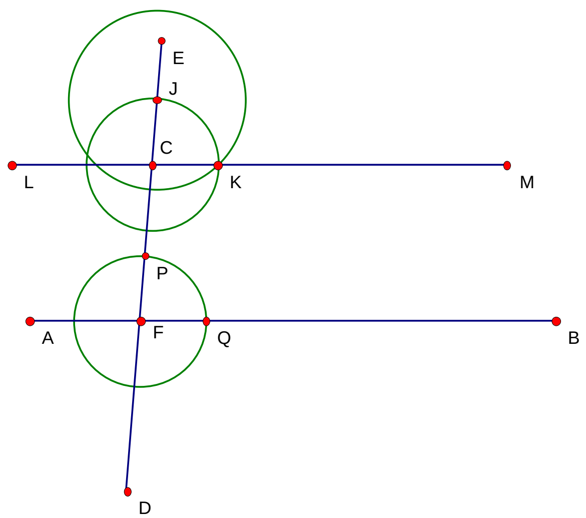
4. Try and prove or explain why this construction produces a perpendicular line.

V. Constructing a line parallel to a given line segment *AB* through a point *C* that is NOT on *AB*.
(Although the picture at right shows full circles, using just the appropriate arcs is sufficient and will make the diagram less crowded.)

1. Draw a segment *DE* through point *C* intersecting *AB* at point *F*.

2. Construct a random circle centered at *F* that intersects *AB* at *Q* and *DE* at *P*. Without changing the compass, draw a circle centered at *C* with the same radius. This circle intersects segment *DE* at point *J*.

3. Draw a circle centered at *J* with a radius equal to the length of *PQ*. This circle intersects the circle centered at *C* at point *K*. The line through *CK* is parallel to original line segment *AB*. Try and prove why this construction works.



Side-Side-Side Triangle Construction

Everyone in the group is going to construct a triangle that has side lengths of 6, 7, and 8 cm. Materials needed are paper, compass, protractor, and ruler.

1. Use the ruler to draw a horizontal segment *AB* that is 8 cm long.

2. With the help of the ruler, determine a radius of 7 cm on the compass, and draw a circle centered at *B* with radius 7 cm.

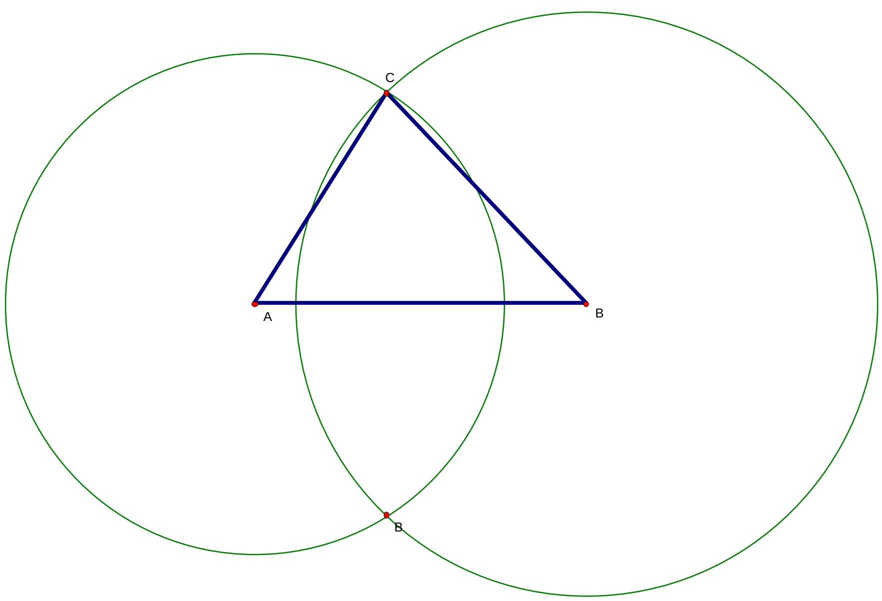
3. Similarly, draw a circle centered at *A* with radius 6 cm.

4. Use either of the two intersection points of the two circles as vertex *C*, and triangle *ABC* has side lengths of 6,7, and 8 cm.

5. Using a protractor, measure as accurately as possible each of the interior angles of the triangle. Write the angle measurements using one-decimal-place accuracy beside the angle itself in the interior of the triangle.

6. Compare your angle measurements with those of the other students. Is the result what you expected? What congruence theorem of triangles does this exercise illustrate?

7. Advanced: Find the exact values of the angles in triangle *ABC*.



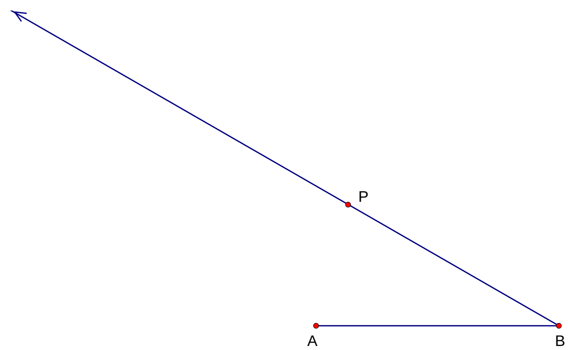
Side-Side-Angle Triangle Construction

Everyone in the group is going to construct a triangle that has side lengths of *CA* = 7 cm, *AB* = 10 cm, and angle *ABC* = 30. Materials needed are graph paper, compass, protractor, and ruler.

1. Use the ruler to draw a horizontal line segment *AB* that is 10 cm long and in the middle of the page.

2. Using the protractor, mark a point *P* so that angle *ABP* is 30 degrees.

3. Using the straight edge, construct ray *BP* so that it extends to the edge of the paper. Your diagram should look like the one at right.



4. The third vertex of the triangle is point *C*, and it is located somewhere on ray *BP*, and exactly 7 cm away from point *A*. Use this information, your compass and ruler to locate point *C*.

5. After locating point *C*, measure the length of the unknown side *BC* in cm, and also measure the unknown angles in degrees.

6. Compare your angle measurements with those of the other students near you. Is the result what you expected? Something very different is going on than in the previous SSS triangle construction activity. Describe in the space below anything interesting you have observed about this SSA construction.

7. If a fellow student were presenting a proof and made a statement that two triangles were congruent by the SSA congruence theorem, what would be your response?

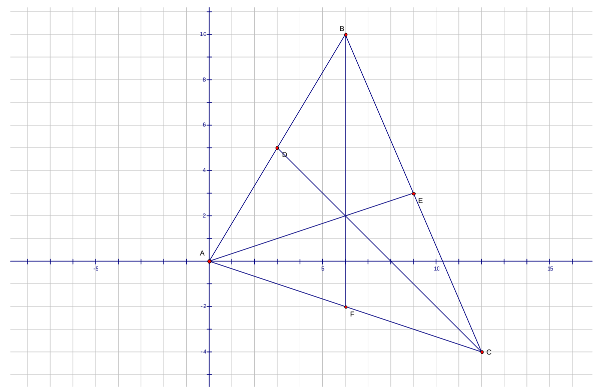
Median Concurrency Investigation

Materials needed are cm cardboard stock graph paper, scissors, calculator, and ruler.

1. Using the cardboard stock cm graph paper, plot any three lattice points (points with integer coordinates) of your choosing to form a triangle *ABC*: Try and make the triangle big.

2. Construct the midpoints of each of the three sides using compass and straight edge techniques from Lab #1, page #3.

3. Use a straight edge to carefully draw in the three *medians* of the triangle. A *median* is the line segment that connects a vertex and its opposite midpoint.



4. It appears that all three medians intersect at the same point. Label this point *G*. This point is called the **centroid** of the triangle. Any point where three or more lines intersect is called a point of *concurrency*.

5. Use scissors to carefully cut out the cardboard triangle. With the sharp end of a pencil or pen, poke a hole all the way through the centroid point *G*.

6. Push a piece of string or twine through the hole and tie a thick enough knot at the end of the string so it cannot be pulled back through the hole at point *G*. Hold up the string at the end opposite the knot so that the triangle is hanging in the air and observe. Write down any observations in the space below.

The Investigation Continued with Coordinates

7. Using a new piece of graph paper, plot the same three points you chose for your triangle in step #1.

8. Plot the midpoints of each of the three sides, and write down the coordinates of the three midpoints in the space below. Remember, the midpoint of (a,b) and (c,d) is the point $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

9. Determine the equation of the three lines that contain each of the medians. Write down the three equations in the space below in *calculator ready point-slope form* $y = m(x-h)+k$.

10. Choose two of the three equations above, and solve simultaneously for their point of intersection, which is the **centroid** of this triangle. Show all your work in the space below. As a check on your work, you may want to graph both lines in your calculator and find their intersection point.

11. To show that the three medians are indeed concurrent, show that the centroid point you found above is on the third line from #9. Show work below. Try and think of more than one way to show that the three medians are indeed concurrent.

12. Look to see how closely the centroid you obtain with algebra matches what you found when constructing the median lines. Record your information in the chart on the next page and also on the board.

13. When you are satisfied that you have correctly found the exact coordinates to your random triangle, add your data to the chart on the board that also appears below for your own notes.

Vertex A	**Vertex B**	**Vertex C**	**Centroid**

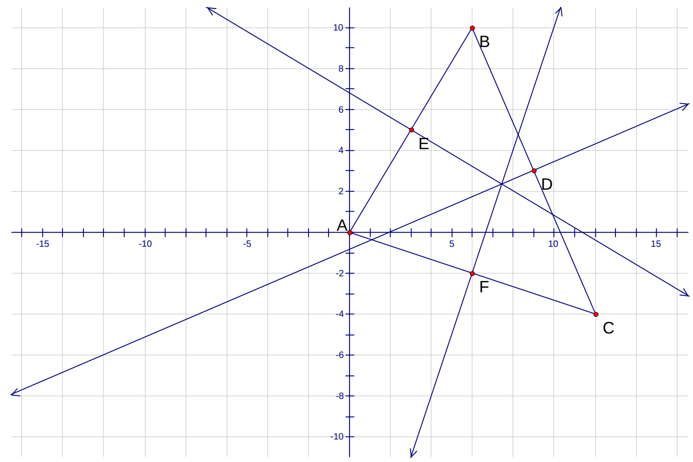
14. Looking at the data in the table above, make a conjecture about the coordinates of the centroid of a triangle and their relationship to the coordinates of the three vertices. Write your conjecture below.

15. Advanced – Prove your conjecture using the constants $A = (0,0)$, $B = (2a, 0)$, and $C = (2b, 2c)$ as coordinates.

Perpendicular Bisector Investigation

Materials needed are graph paper, compass, calculator, and ruler.

1. Using a new piece of cm graph paper, plot the points $A = (0,0)$, $B = (6,10)$, and $C = (12,-4)$.



2. Using the compass and straight edge construction techniques practiced earlier, construct the perpendicular bisectors of each of the sides of the triangle *ABC*.

3. It appears that the three perpendicular bisectors all intersect at the same point. Using a ruler, measure as accurately as possible the distance in cm from this point to each of the vertices. What do you notice?

4. If these distances are indeed the same, then a circle could be constructed with its center at this point so that the circle passes through each of the vertices of the triangle. This circle is called the ***circumcircle*** of the triangle, and the center point is called the ***circumcenter***. Use a compass to construct this circle.

5. Let's calculate the coordinates of the circumcenter of this triangle exactly by using coordinate geometry. Remember, the midpoint of (a,b) and (c,d) is the point $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$. Also recall that the slopes of perpendicular lines are negative reciprocals of each other. In other words, since the slope of side AC is $-1/3$, then the slope of the perpendicular bisector must be 3. Write down the equations of the three perpendicular bisectors in the space below using *calculator ready point-slope form* $y = m(x-h)+k$.

6. Choose two of the three equations above, and solve simultaneously for their point of intersection, which is the **circumcenter** of this triangle. Show all your work in the space below. As a check on your work, you may want to graph both lines in your calculator and find their intersection point.

7. To prove that the three perpendicular bisectors above are indeed concurrent, show that the circumcenter point you found is indeed on the third line from #5. Show work below.

8. When you reach this point, go to the teacher to be assigned one of the triangles from the chart on the next page. Once assigned a specific triangle, do the following:
   a) Plot the points and draw an accurate sketch of the triangle
   b) Identify whether your triangle is obtuse, acute, or right.
   c) Repeat steps #5-#7 to find the circumcenter and to verify it is the point of concurrency of the three perpendicular bisectors of your triangle. Use the space below to show your work.
   d) Determine if your circumcenter is inside, outside, or exactly on your triangle.
   e) Put your information on the chart on the blackboard.

9. When you are satisfied that you have correctly found the exact coordinates to your triangle, add your data to the chart on the board that also appears below for your own notes.

Vertex A	Vertex B	Vertex C	Obtuse, Right, or Acute	Circumcenter	Radius of Circumcircle	Inside , Outside , or On the triangle
(6,3)	(8,−1)	(2,−9)				
(4,8)	(7,3)	(−9,−5)				
(7,5)	(10,0)	(−6,−8)				
(−4,2)	(0,8)	(2,−2)				
(14,10)	(2,2)	(9,−15)				
(11,8)	(9,−2)	(−1,0)				
(4,16)	(9,1)	(5,−7)				
(−11,4)	(10,7)	(2,−9)				
(−5,−8)	(15,2)	(13,10)				
(−3,12)	(9,2)	(−7,2)				
(−8,−5)	(13,10)	(13,−12)				
(−17,2)	(19,10)	(15,−6)				

10. Looking at the data in the table above, make a conjecture about location of the circumcenter as determined by the type of triangle.

11. In this lab, you have now learned that the circumcenter is the unique point that is the point of concurrency of the perpendicular bisectors of a triangle. Therefore, it is equidistant from the three vertices in a triangle, and thus the center of the circumcircle that circumscribes the triangle. Let's try and find the equation of the circumcircle for the first triangle in the chart on the previous page. Let $A = (0,0)$, $B = (4,5)$, and $C = (10,-8)$. Recall, that the circumcenter O is at $(7,-1.5)$. Let $P = (x, y)$ represent any the point on the circumcircle. Write down in the space below the distance expression from $P = (x,y)$ to $O = (7, -1.5)$. Recall that the distance formula between two points (a,b) and (c,d) is $\sqrt{(a-c)^2+(b-d)^2}$.

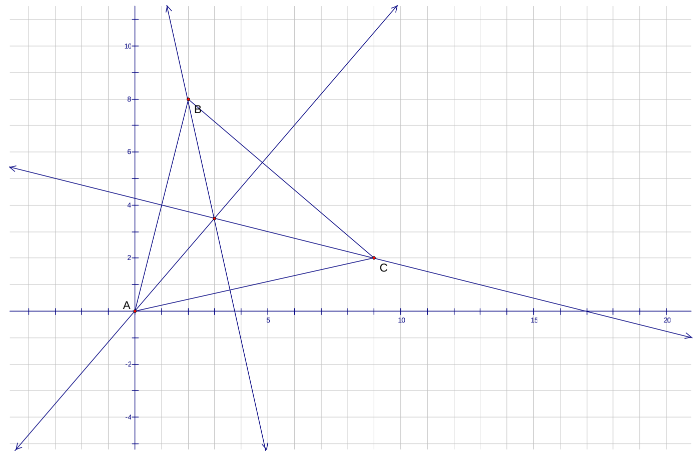
12. The equation in #11 above does describe all the points that are on the circumscribed circle, but it is not in the form that circles are normally expressed. By squaring both sides, you should get an equation that has no radicals in it, and looks like the one below. This is the typical form that one leaves a circle. This is called ***center-radius form*** of a circle and is generalized $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius. What is the radius of the circle described below, which is the circumcircle of the triangle ABC?

$$(x-7)^2+(y+1.5)^2=51.25$$

13. Go back to the triangle you were assigned from the chart on the previous page, and determine the equation of the circumcircle of that triangle in ***center-radius form***. Write the equation in the space below.

Altitude Investigation

Materials needed are graph paper, colored pencils, compass, protractor, calculator, and ruler.



1. Using a new piece of cm graph paper, plot the points $A = (0,0)$, $B = (2,8)$, and $C = (9,2)$.

2. Using the compass and straight edge construction techniques practiced earlier, construct the line perpendicular to BC that contains point A. Repeat this construction for the line perpendicular to AB that contains point C, and the line perpendicular to AC that contains point B. (try and use lightly drawn construction arcs in order to not clutter the drawing too much)

3. It appears that the three lines perpendicular to the sides of the triangle and passing through the opposite vertex all intersect at the same point. Estimate the coordinates of this point.

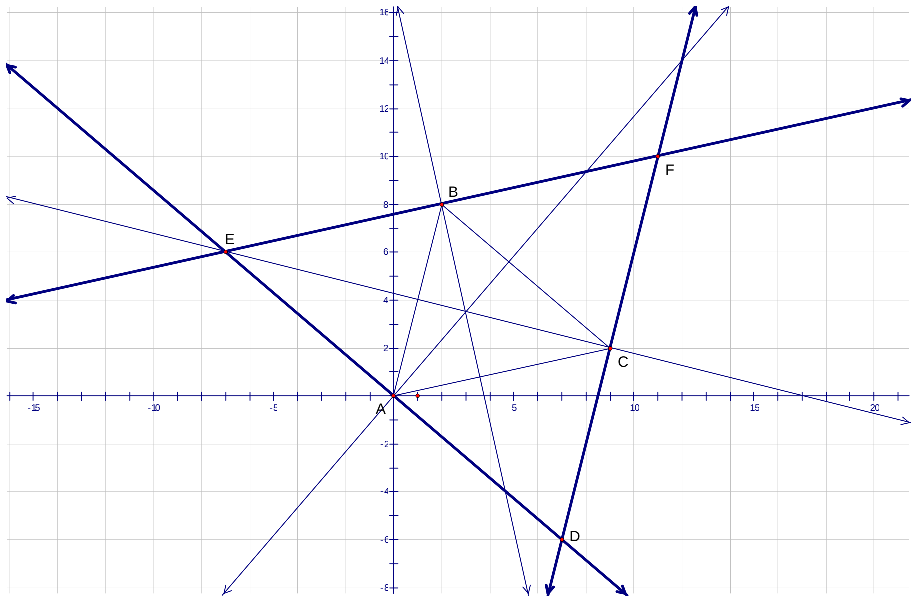
4. In order to verify that these three lines are indeed ***concurrent*** (passing through the same point), let's do the problem algebraically. This will involve writing down the equations of the three perpendicular lines in the space below using *calculator ready point-slope form* $y = m(x-h)+k$.

   Line through A: Line through B: Line through C:

5. To check for concurrency, you will need to find the intersection of two of the lines, and then see if that point of intersection is indeed on the third line. In the space below, find the intersection of the lines through A and B respectively by solving their equations simultaneously. When you get your answer, check back to see if it makes sense relative to your estimate in step #3.

6. Now, simply substitute the coordinates for the point of intersection back into the equation of the third line to verify that the three lines are indeed concurrent.

7. Is there something special about this particular triangle, or are the three lines through the vertices of a triangle to the opposite side always concurrent? Let's draw an interesting triangle using our original triangle that will help answer this question. To save time, do not use construction techniques, but draw the following lines with care. Draw a parallel line to side *BC* that contains point *A*. Similarly, draw a parallel line to *AB* containing point *C*, and then a parallel line to *AC* containing point *B*. Your sketch should look like the one at right. Label the intersection points *D.E*, and *F* as shown in the diagram.



8. In the space below, give a convincing argument that each of the following quadrilaterals is a parallelogram: *ABFC*, *AEBC*, and *ABCD*. It may help to outline the quadrilaterals in different colors to make them easy to see.

9. a) In the space below, explain why *EB* = *BF, EA* = *AD,* and *DC* = *CF.*

   b) What does this say about points *A*,*B*, and *C* with respect to triangle *DEF*?

   c) What does this say about the perpendicular lines through *A*,*B*, and *C* with respect to triangle *DEF*?

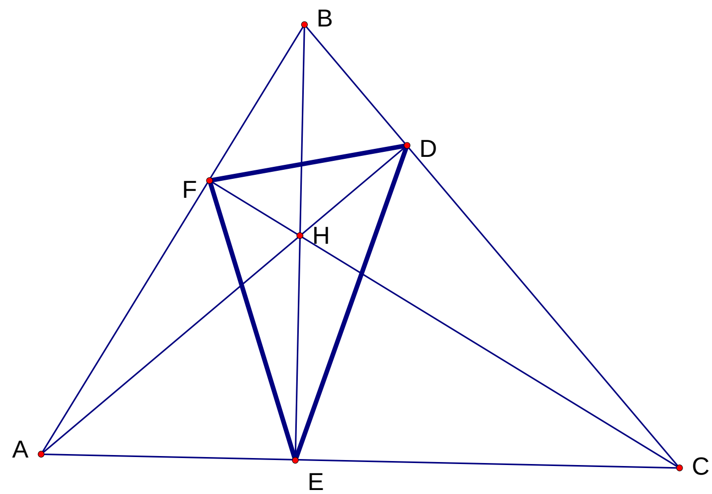
10. Now that you have concluded above that each of the three perpendicular lines through *A*,*B*, and *C* is a **perpendicular bisector of triangle *DEF***, what is their common intersection point called with respect to triangle *DEF*?

11. If you said ***circumcenter*** of triangle *DEF*, then you are a star. Since there was nothing specific to our particular triangle in steps #7 - #9, then you have proven that the three lines perpendicular to the sides of a triangle and passing through the opposite vertex all intersect at the same point. This point of concurrency is called the ***orthocenter*** of the original triangle *ABC*. ABC is called the ***medial triangle*** of triangle *DEF*. In summary, you have proven the circumcenter of any triangle is also the orthocenter of its medial triangle.

More about the Orthocenter for Advanced Students

12. Start with a new piece of paper and draw a large scalene triangle labeled *ABC*. There will be no coordinate geometry in this part of the problem, so do not worry about that. Once the triangle has been drawn, construct the altitudes to the three sides of the triangle. If you want to save time, **accurately** draw the altitudes instead of constructing them. Point *D* should be the label of the intersection of side *BC* with the altitude from vertex *A*. Let *E* be the foot of the perpendicular opposite *B*, and F the foot of the perpendicular opposite vertex *C*. Also, label the orthocenter, the intersection point of the three altitudes, point *H*.

13. Connect points *D, E*, and *F* to form a triangle that is called the ***orthic triangle*** of *ABC*. Your sketch should now look like the one at right.

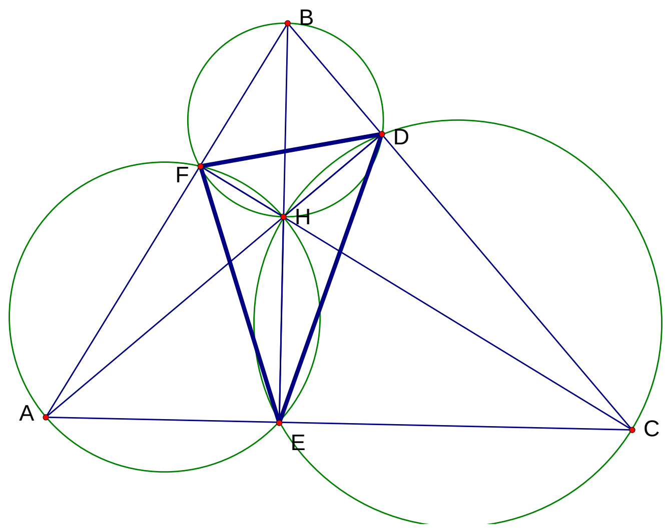


14. Using a protractor, measure the following angles and record them below.
ADF *ADE* *BEF* *BED* *CFE* *CFD.*

15. Do you notice anything interesting about the angle measurements? Hopefully, you have done the drawing and measuring accurately enough to notice that it appears that *AD, BE*, and *CF* are angle bisectors of triangle *DEF*. Let's see if we can prove this fact.

16. Recall an important fact from geometry about quadrilaterals. A quadrilateral is ***cyclic*** (it can be inscribed in a circle), if and only if its opposite angles are supplementary. Explain in the space below why *AFHE*, *CEHD*, and *BDHF* are all cyclic quadrilaterals.

17. Using a compass, draw the three circles that circumscribe quadrilaterals *AFHE*, *CEHD*, and *BDHF*. You can reasonably estimate where each of the centers of the circles are located using perpendicular bisectors of two of the sides of a quadrilateral. It is recommended that you draw the circles in lightly so the drawing does not look too cluttered. When finished, your sketch should look like the one at right.



18. First, it is important to note the following angle equivalencies:
$\angle BCF = \angle BAD$ and $\angle ABE = \angle ACF$ and $\angle CBE = \angle CAD$
You can measure with your protractor to verify, but also write the mathematical reason in the space below.

19. We are almost there, so keep concentrating. Our goal is to show that *AD, BE*, and *CF* are angle bisectors of triangle *DEF*. Notice that equal angles *BCF* and *BAD* intersect arcs on two different circles. *BCF* intersects arc *HD* on the circle containing vertex *C*, and *BAD* intersects arc *HF* on the circle containing vertex *A*. This means that the arc angles for arc *HD* and arc *HF* are equal. Now, look carefully at angles *BED* and *BEF*. What do you notice about them?

20. Finish off the rest of the details below as to why *AD* and *CF* are also angle bisectors. This result means that **the orthocenter of triangle *ABC* is also the incenter of its orthic triangle *DEF***. This result is an alternative proof that the altitudes of a triangle are concurrent.

The Euler Line

Materials needed are graph paper, calculator, and ruler.

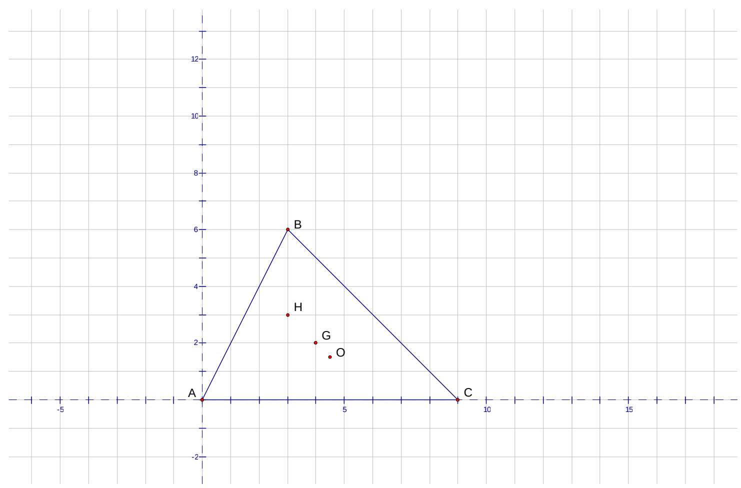
1. Let's start with drawing the triangle $A = (0,0)$, $B = (3,6)$, and $C = (9,0)$ on the graph paper. Recall the definitions of the **Centroid**, **Circumcenter**, and **Orthocenter** from our previous work, and write them in the space below.

   Centroid (G):

   Circumcenter (O):

   Orthocenter (H):

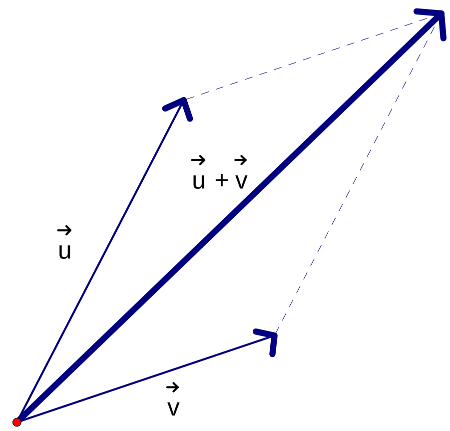
2. Use the definitions above and algebra to determine the coordinates of each of these three special points of concurrency for the given triangle. Try and take advantage that one of the sides of the triangle is horizontal. Show all your work below, and plot the three points carefully in your sketch using the labels *G*, *O*, and *H* as indicated above. When finished, your sketch should like the one at right.



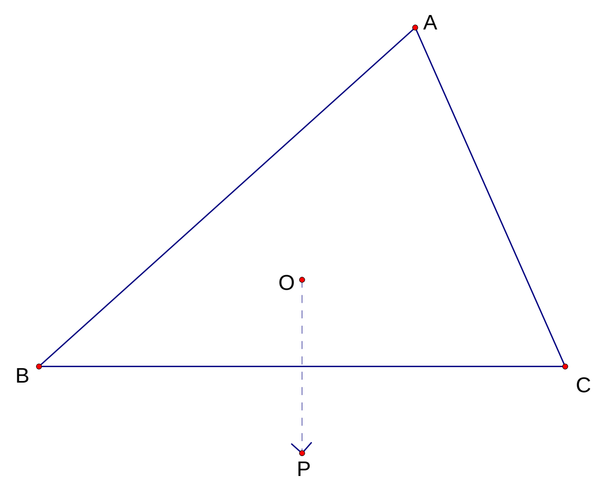
3. It should appear that the Centroid, Circumcenter, and Orthocenter are all on the same line. Test out this conjecture by doing appropriate work in the space below.

4. Hopefully, you discovered that the three points are $(4,2)$, $(4.5,1.5)$, and $(3,3)$, and that they are indeed collinear on a line with a slope of -1. If not, go back and try and find where you made your mistake. It should also appear that the distance from *G* to *H* is about twice as much as the distance from *G* to *O*. Check out this conjecture, and show your work below.

5. Now the interesting question to pursue becomes whether or not the collinear relationship and distance relationship between these three points is always true for all triangles. First, recall this little reminder about vectors. Vectors are added by the "foot to head" method illustrated at right. The sum of two vectors $\vec{u}+\vec{v}$ thus becomes the diagonal of a parallelogram that has the two vectors being added as adjacent sides.

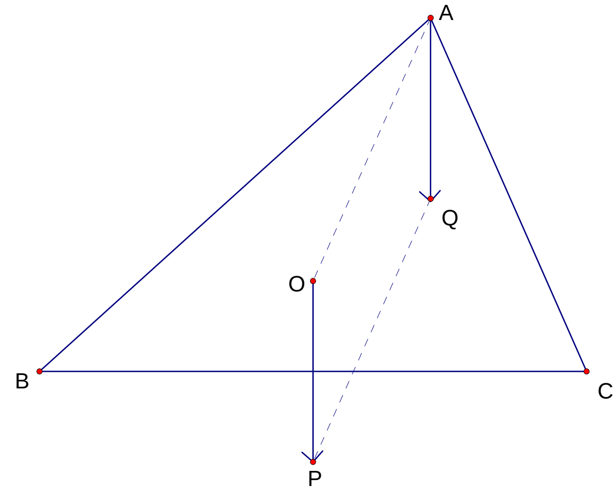


6. Start with a blank piece of paper and draw a triangle *ABC* similar in shape to the one shown at right. Being reasonable accurate, locate the ***circumcenter*** of this triangle, and label it *O*. Explain below why the vector $\overrightarrow{OB}+\overrightarrow{OC}$ is a vector that is perpendicular to side *BC* of the triangle. It may help to carefully draw in the vectors and even the parallelogram. Call this vector $\overrightarrow{OP}$.



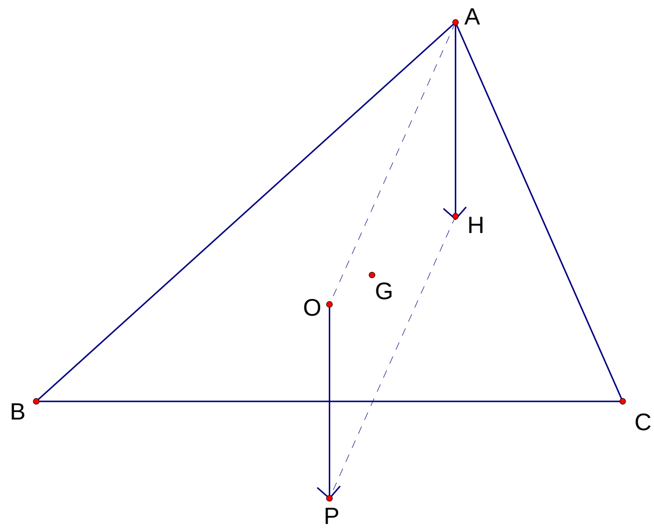
7. Now, look at the vector $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OP}$, and explain why *Q* is a point that is on the altitude from vertex *A*. Again, it may be helpful to carefully draw $\overrightarrow{OQ}$.

8. Similary to steps #6 and #7, first show that $\overrightarrow{OA} + \overrightarrow{OC}$ is a vector that is perpendicular to side *AC* of the triangle, and then explain/show why $\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$ is also perpendicular to side *AC*. However, $\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$ is exactly what we had on the right side in #7 when we stated that $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OP}$. This means that point *Q* is not only on the altitude from vertex *A* as pointed out in #7, but it is also on the altitude from vertex *B* from the work in #8. What important conclusion can you state about what special point *Q* must be.



9. If you stated that point *Q* must be the same as the **orthocenter *H*** of triangle *ABC*, you are a superstar. It is the only point that is on both the altitude from *A* and the altitude from *B*.

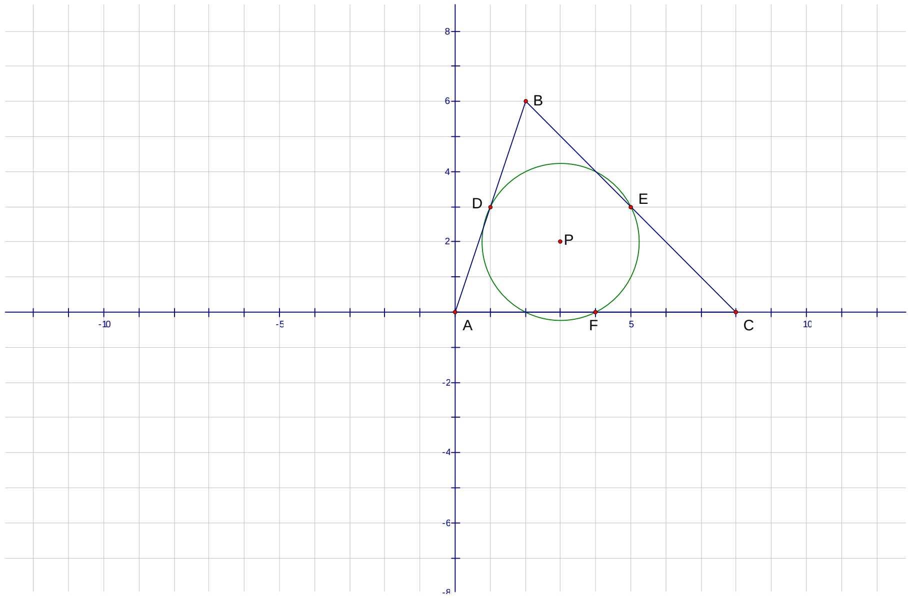
10. Now comes the slick part. Recall from your earlier work that the **centroid *G*** of a triangle can be found 2/3 of the way along the median from any vertex to the opposite midpoint. The vector $\overrightarrow{BA} + \overrightarrow{BC}$ is the diagonal of the parallelogram with *BA* and *BC* as adjacent sides. This diagonal is exactly twice the median from *B*, so 2/3 of the median is $\frac{1}{3}\left(\overrightarrow{BA} + \overrightarrow{BC}\right)$. Without coordinates, the vector way of saying the same thing is **centroid *G*** = $\overrightarrow{OB} + \frac{1}{3}\left(\overrightarrow{BA} + \overrightarrow{BC}\right)$ for any point *O* in the plane. Show that this is equivalent to saying that **centroid *G*** = $\frac{1}{3}\left(\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}\right) = \frac{1}{3}\overrightarrow{OH}$. Try and convince yourself that this makes sense. Now the proof that ***O*,*G***, and ***H*** are collinear with the distance relationship $|GH| = 2|GO|$ is complete. Locate point *G* in your diagram, and the final picture should look like the one at right.



The "9-Point" Circle
Sometimes called Euler's Circle or the Feuerbach Circle

Materials needed are graph paper, compass, and ruler.

1. Let's start with drawing the triangle $A = (0,0)$, $B = (2,6)$, and $C = (8,0)$ on the graph paper. Make the scale two graph paper lines per unit in order to have a good size triangle. Determine the coordinates of the three midpoints of this triangle and label them D,E, and F . In the space below, determine the ***circumcenter*** of triangle DEF, and plot this point carefully. Then use the compass to draw the circle that contains the three midpoints D, E, and F of the original triangle. When finished, your sketch should look like the one at right. Make sure you note the center of the coordinates of the center of the circle P, and the radius of the circle.

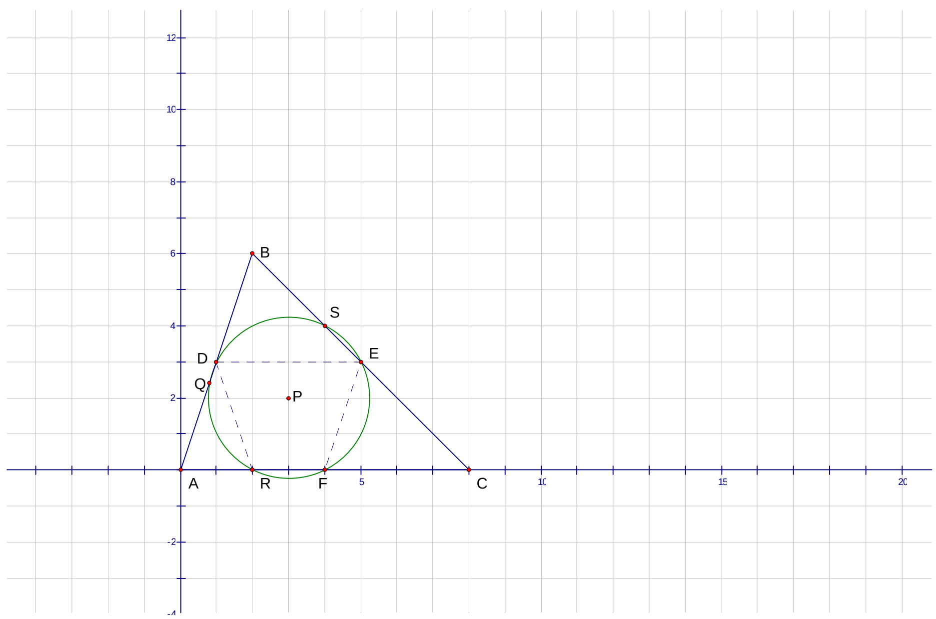


$D =$ $E =$ $F =$ $P =$

2. Next, determine the coordinates for the three ***"feet"*** of the altitudes for triangle *ABC*. This means, the three points that the altitudes intersect the opposite side. Show all your work below, label the three *feet* *Q*, *R*, and *S*, record the coordinates, and plot carefully on your sketch.

Q = *R* = *S* =

3. Using the center of the circle *P* from step #1, determine the distance from *P* to *Q*, *R*, and *S*. What do you notice?



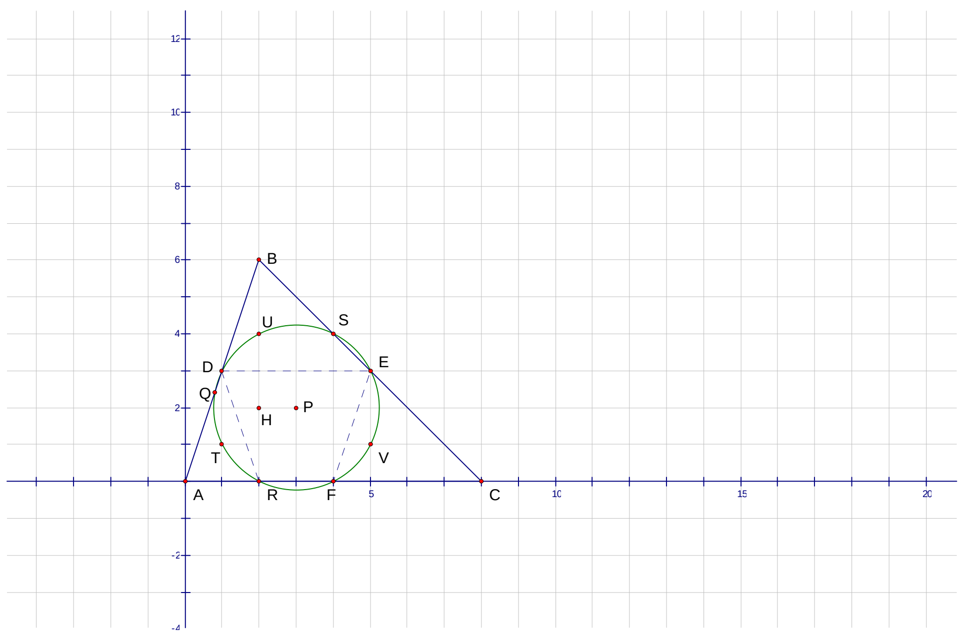
If you said that the three *feet* *Q*, *R*, and *S* lie on the same circle as *D*, *E*, and *F*, then your calculations have been perfect. If this is not the case, go back and look at your work and see if you can find your mistake. In order to see why point *R* is on the same circle as *D*, *E*, and *F*, draw the quadrilateral *DEFR* and identify what type of quadrilateral it is. Your sketch should look like the one above.

4. If you thought the quadrilateral *DEFR* was an isosceles trapezoid, give yourself a pat on the back. Hopefully you have done some analysis to verify this conjecture and not relied on the visual. If not, do it now. An important theorem about quadrilaterals is "a quadrilateral is ***"cyclic"*** (can be circumscribed by a circle) if their opposite angles are supplementary." Try and explain below why the opposite angles of an isosceles trapezoid are supplementary, and thus it is cyclic. The consequence of this is that *R* must lie on the same circle as *D, E,* and *F*.

5. Name the isosceles trapezoids that assure that *Q* and *S* also lie on the circle centered at *P* and containing *D, E,* and *F*.

6. We now have six points all on the same circle, and from the title of this worksheet, we know we have three more to find. It turns out that an important point to find that will lead to the remaining three points is our old friend the **"orthocenter" H**. This should not be too hard since you already have the equations for the altitudes from step #2. Use two of them to find the orthocenter, and plot it carefully. Show your work below.

7. It should be obvious that the orthocenter *H* itself is not on the circle, but amazingly, the three midpoints of the segments connecting the orthocenter to each of the original vertices *A, B,* and *C* appears to land right on the circle. Find the coordinates of these three midpoints, label them *T, U, V*, and verify that they are indeed the same $\sqrt{5}$ units away from point *P* as the other six points. Now we have the "9-Point Circle" for this triangle.



8. To see why these midpoints are on the same circle as the other six, let's take a close look at one of those points. Point *U* is the midpoint of *H* and *B*. Point *D* is the midpoint of *AB*. Why is *DU* parallel to the altitude *AS*?

9. Why is *FD* parallel to side *BC*?

   Why is *FD* perpendicular to altitude *AS*?

   Now look closely at triangle *FDU*. Why is *FDU* a right triangle with a right angle at *D*?

10. Using the important theorem that a right triangle's circumcenter is the midpoint of its hypotenuse, what has to be the center of the circle that circumscribes *FDU*?

11. If you said the midpoint of hypotenuse *UF*, you are doing great. Since we already know that *P* is the center of the circle containing *F* and *D*, then *P* must also be the midpoint of *F* and *U*. So, right triangle *FDU* is the key for knowing *U* is on the same circle as *F* and *D*. Name the right triangles below that are the keys to why *T* and *V* are also on this circle.

12. Everything we have done in this worksheet has involved a specific triangle with fixed coordinates. Start fresh with a brand new large triangle with no coordinates, and construct the *9-point circle*. If you need to conserve time, you can shortcut actual construction techniques for midpoints and perpendiculars. Use what you have done in steps #1 through #7 as a guide. Make sure you understand how to prove that these nine points are indeed on the same circle.

An Area Investigation

Materials needed are cm graph paper, calculator, and ruler.

Problem: Find the area of triangle ABC where $A = (0,0)$, $B = (2,6)$, and $C = (8,2)$.

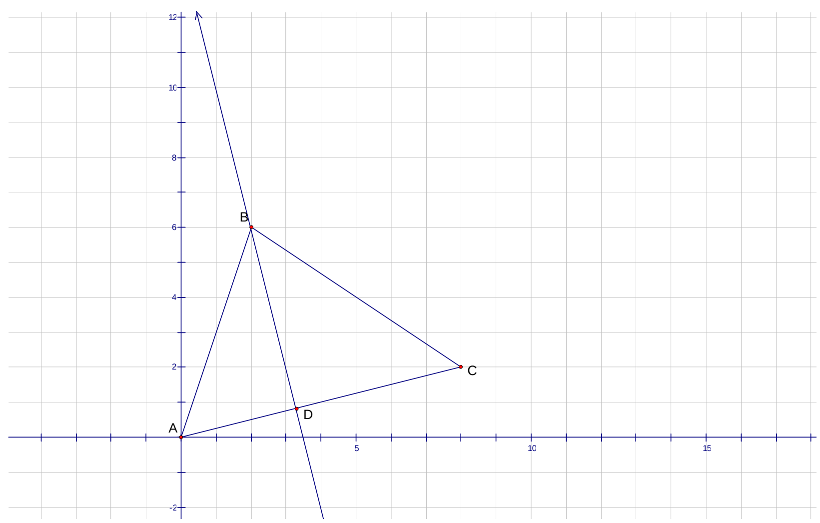
1. Draw the triangle on a piece of graph paper using a scale of one grid lines per unit.

2. Make a rough estimate of the area by counting the number of unit blocks within the triangle.

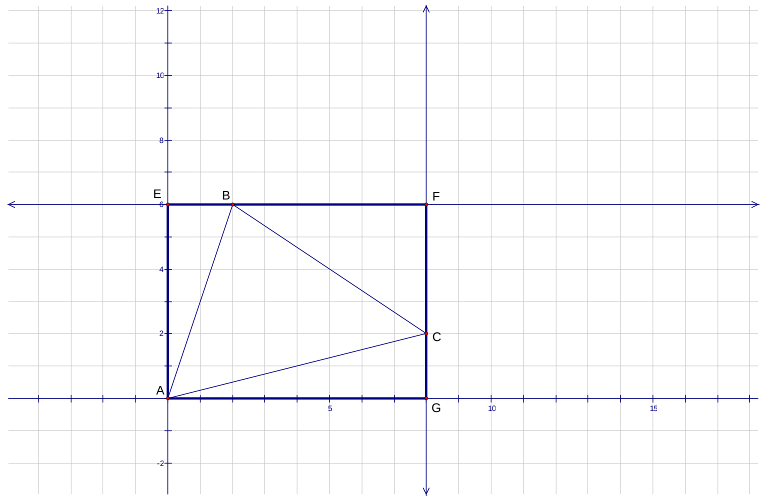
3. Also make an estimate by measuring with a ruler one of the side lengths and the altitude to that side. Show your measurements and calculations below.

4. To find the exact area of this triangle algebraically is not an easy task. To emphasize this point, let's give it a try.
   a) Write the equation of the line perpendicular to side AC containing vertex B.

   b) Find the intersection point D of the line above with side AC. Use the length of line segment BD and the length of side AC to determine the area of the triangle. Is it reasonable close to your estimates from lines 2 and 3.



5. Let's try another approach. Box in triangle *ABC* by drawing a horizontal line through vertex *B*, a vertical line through vertex *C*, and using the coordinate axis to form a rectangular box *EFGA* that encloses the triangle. Can you know see an easy way to figure out the exact area of triangle *ABC* by cleverly using the area of the rectangle *EFGA*, and also your knowledge of right triangle areas. Do your work below, and compare your answer to previous results.



6. Now that we have done an example, it should be easy to replicate this technique on some other triangles. Find the area of the triangles below whose vertices are given.
   a) (1,0), (−3,6), (7,4)

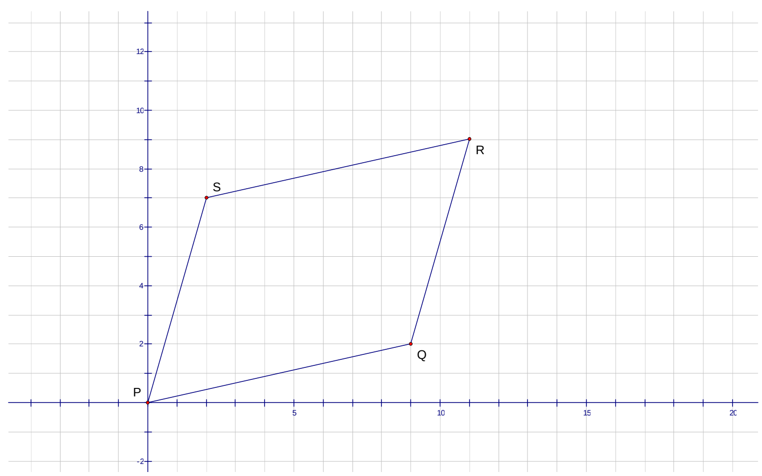
   b) (−2,−3), (0,4), (5,−1)

   c) (0,0), (−2,4), (6,−1) Pay special attention to this triangle, and note a slight adjustment in your technique. What made this different?

7. Clearly the above technique of boxing in a triangle and determining its area by using rectangle and right triangle areas can be used with no limitations on the coordinates of the vertices. However, if the coordinates of the vertices happen to be integers, what can you definitively about the area of the triangle, and why?

8. What if the figure in question was a parallelogram? Would the same technique work? Let's try it on the parallelogram with vertices (0,0), (7,2), (8,5), and (1,3).

9. The figure at right shows a parallelogram *PQRS*, three of whose vertices are $P = (0,0)$, $Q = (a,b)$, and $S = (c,d)$.
   a) Find the coordinates of *R*.
   b) Use the box technique to find a formula for the area of *PQRS* in simplified form.

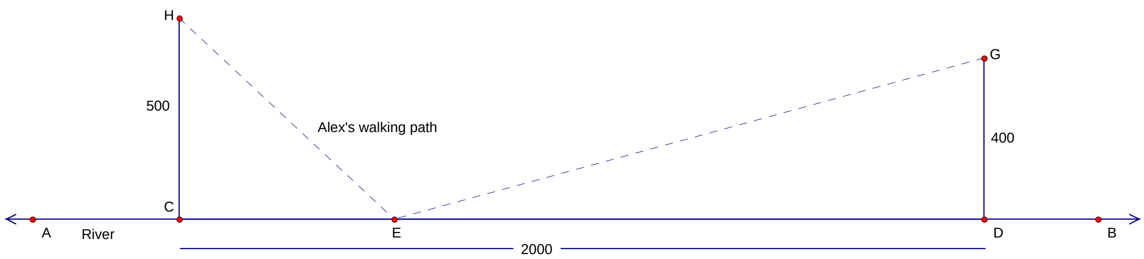


10. What would be the formula for the area of triangle *PQS*?

11. How could you use the formulas you derived to find the area of triangles and parallelograms that did not have a vertex at the origin? Once you have figured this out, use the formula to find the area of the parallelogram with vertices at (2,5), (7,6), (10,10), and (5,9).

A Minimum Distance Investigation

Alex needs to walk from his house at H, down to the river represented by line AB, fill up some water bottles, and then bring them to his grandmother's house at G. The shortest distance from his house to the river is at point C, 500 meters from H, and the shortest distance from his grandmother's house to the river is at point D, 400 meters from G. Points C and D are 2000 meters apart. If Alex travels to point E on the river to fill up the water bottles, how far from C should E be located in order to travel the least distance possible for the entire trip?



1) Using a scaled model on graph paper, a ruler, and a protractor, complete the chart below.

Distance from C to E (m)	Measured distance HE (cm)	Measured distance EG (cm)	Total measured distance (cm)	Measure of angles HEC and GED
0				
200				
400				
600				
800				
1000				
1200				
1400				
1600				
1800				
2000				

2) Enter the data from column 1 and column 4 in the above table in the TI calculator and make a scatter plot of the data.

3) What conclusions, (if any), can you make so far?

4) Using the Pythagorean Theorem (or the distance formula), fill in the chart below with calculated distances rather than measured distances.

Distance from C to E (m)	Calculated distance HE (m)	Calculated distance EG (m)	Total calculated distance (m)	Calculated angles HEC and GED by trigonometry
0				
200				
400				
600				
800				
1000				
1200				
1400				
1600				
1800				
2000				

5) Enter the data from column 1 and column 4 in the above table in the TI calculator and make a scatter plot of the data. How does this compare with the previous plot? Are your conclusions, (if any), still supported by the data?

6) Add the following row to the table above and fill in the empty columns.

Distance from C to E (m)	Calculated distance HE (m)	Calculated distance EG (m)	Total calculated distance (m)
x			

7) Let y_1 = the expression in column 4 above, and graph on the TI calculator. How is this curve related to the previous scatter plot?

8) Use the calculator to find the minimum of the distance function in y_1. Does this agree with you're earlier work?

9) Using the scaled model on graph paper, reflect point *G* about line *AB* (the *x*-axis), and call it *K*. See if you can come up with another way to determine the shortest distance from *H* to line *AB*, and then to *G*.

Advanced

10) What would happen to the optimal landing point *E* if the speeds Alex travels to and from the river were not the same? Of course, by optimal, we now must refer to **minimizing total time** rather than total distance. Let's look at an example where the speed to the river is 5 m/s and the speed to grandma's house after filling up the bottles is 4 m/s. Verify that the information in the chart below makes sense.

Distance from C to E (m)	Time from H to E in sec	Time from E to G in sec	Total time in seconds
x	$\dfrac{\sqrt{500^2+x^2}}{5}$	$\dfrac{\sqrt{400^2+(2000-x)^2}}{4}$	Sum of the previous two columns.

11) After entering the total time into y_1 in the calculator, find the value of *x* that minimizes the function, as well as the minimum amount of time. If all goes well, you should have found an optimal value for *x* of 1531.53, and a minimal time of 476.22 seconds. Explain in the space below why it makes sense that the optimal *x* value is now greater then the situation when the speeds were the same (which is equivalent to just minimizing distance).

12) When you have finished step #11, go to the instructor and get assigned one of the new rows in the chart below. Complete the information in your row, and also enter it on the board.

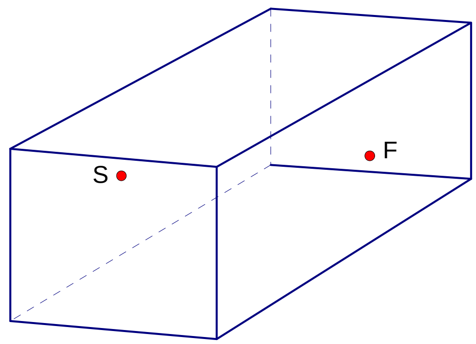
Speed from *H* to the River	Speed from the river to *G*	Distance from *C* to optimal point *E* that minimizes time.	$\dfrac{CE}{HE}$	$\dfrac{ED}{GD}$
5	4	1531.5	0.95	0.76
6	4			
6	3			
5	2			
6	2			
4	5			
4	6			
3	6			
2	5			
2	6			

**There is an important relationship between the first 2 columns and the last 2 columns. Find it.

The Spider – Fly Problem

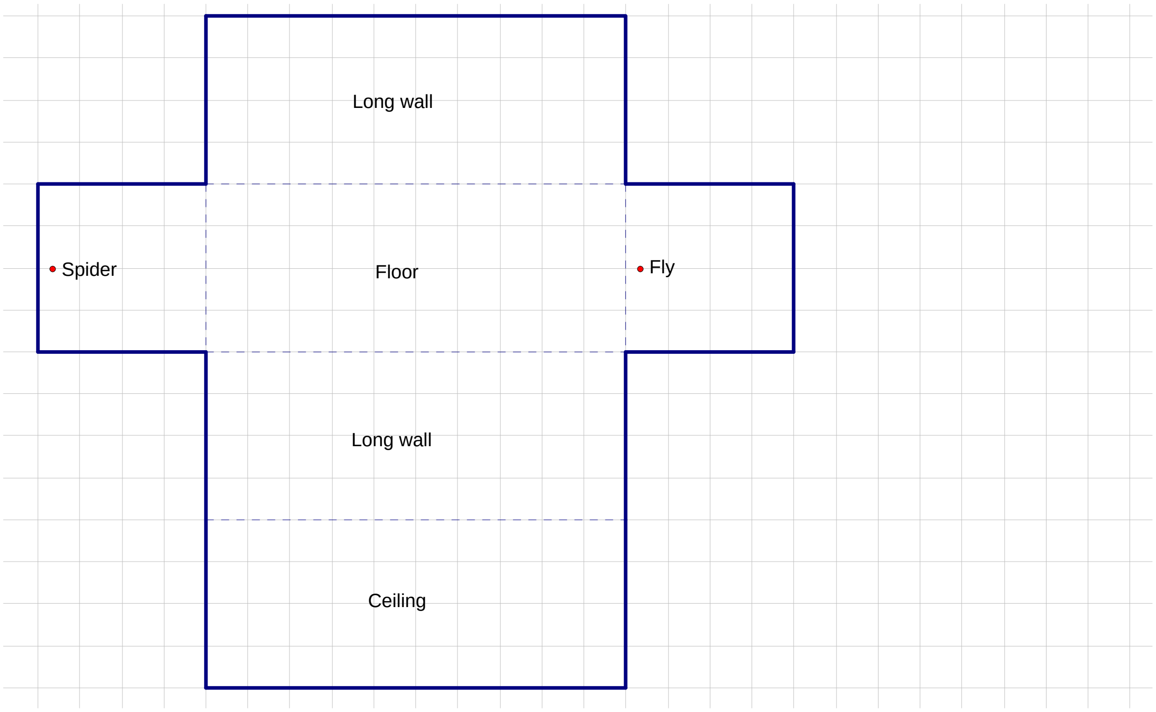
Note: Do Math 2 17/11 as a preliminary exercise to this activity.

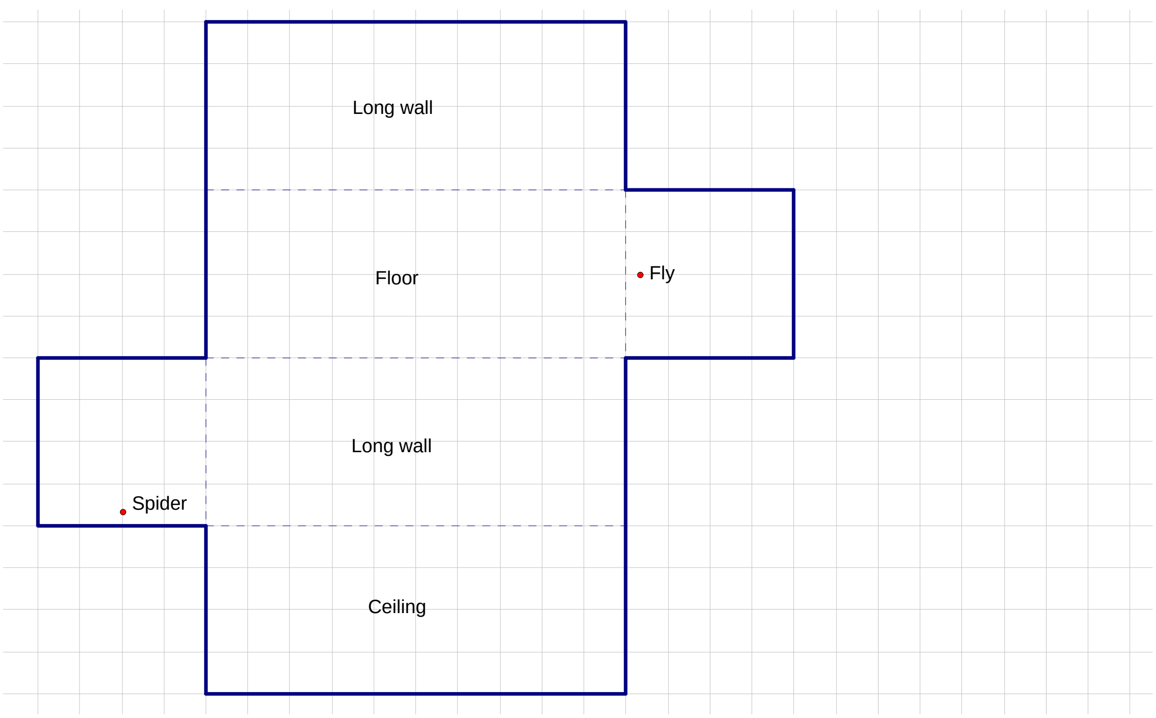
A spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the top ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider have to crawl?



1) On page two, there are two different possible layouts that can be cut out and folded into a rectangular box to simulate the situation described above. There are other possibilities as well. Use these layouts to help you find possible routes from the spider to the fly. Do they both result in the same distance traveled by the spider?

2) Use blank graph paper to construct other possible layouts of this rectangular box that will enable you to solve the problem above. Write up and show your results in the space below. Good luck

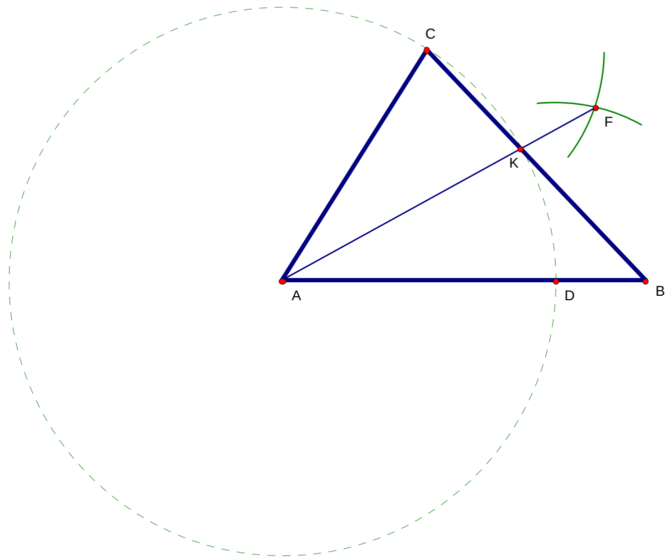

Long wall
Spider
Floor
Fly
Long wall
Ceiling


Long wall
Floor
Fly
Long wall
Spider
Ceiling

Angle Bisector Investigation

Materials needed are paper, compass, and ruler.

1. Below is the picture of the 6-7-8 triangle constructed earlier with side *AB* = 8, *BC* = 7, and *AC* = 6.
2. Apply the angle bisector construction technique to construct the angle bisector of angle opposite the 7 side in your 6-7-8 triangle. Mark the point of intersection between the angle bisector and the opposite side as point *K*.
3. Using a ruler, measure as accurately as possible the lengths of *CK* and *BK*.
4. Is point *K* the midpoint of side *BC*? If not, which side is it closer to? Do you notice anything interesting about the location of point *K* relative to *B* and *C*?
5. Everyone in the group is to construct a scalene triangle *ABC* that has three different integer side lengths from the table below. See the instructor to be assigned one of these triangles.
6. Repeat steps #1- #4 bisecting angle *BAC* in your new triangle. Once again, measure accurately the lengths *BK* and *CK* created on the opposite side. As a class, fill in the chart below and try and see if there is anything going on that we could conjecture about yet? Perhaps another column of information might prove useful?



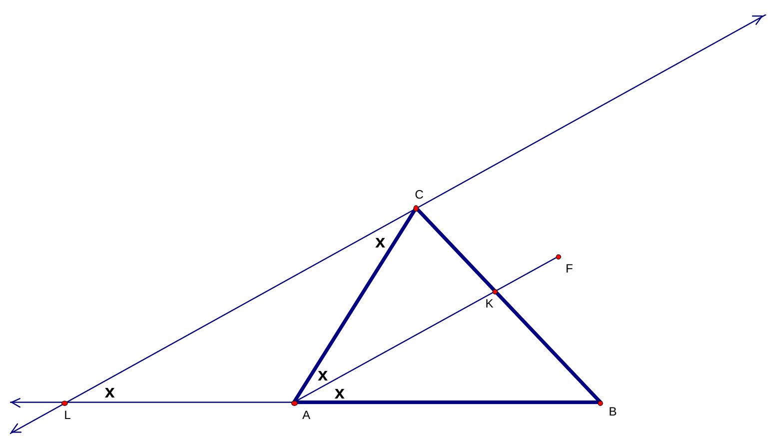
Side Length *AB*	Side Length *AC*	Side Length *BC*	Length *BK*	Length *CK*
8	6	7		
9	6	5		
10	6	8		
12	6	9		
6	4	5		
10	8	9		
7	5	6		
10	5	6		
4	8	6		
6	9	10		
4	5	6		
6	8	10		
8	12	10		
6	6	10		
15	9	12		

7. Let's do one of these with coordinates and gather even more accurate data. Plot the points $A = (0,0)$, $B = (6,0)$, and $C = (0,8)$.

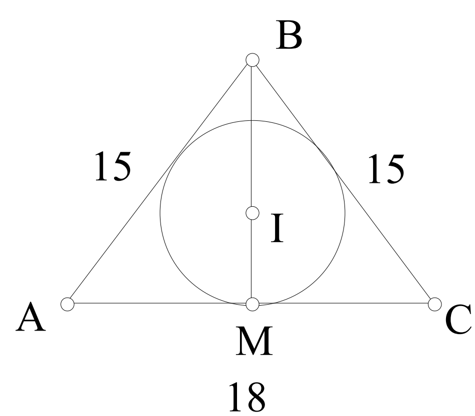
8. We know the angle bisector of angle BAC is the line $y = x$. Determine algebraically where $y = x$ intersects the opposite side. You will need the equation of the line containing side BC. Show all work below.

9. Calculate using the distance formula the **exact** lengths created on the opposite side and see if any pattern from your previous work still holds true. If so, try and clearly state that conjecture in the space below:

10. The diagram below where the line CL is constructed parallel to the angle bisector AK can be used to prove a very useful theorem about angle bisectors that perhaps you had as a conjecture above.



11. Use the angle bisector theorem to find the length IM in the diagram to the right. I is the incenter of the 15,15,18 isosceles triangle. This problem can be solved using a couple of different methods as a way to check your work.

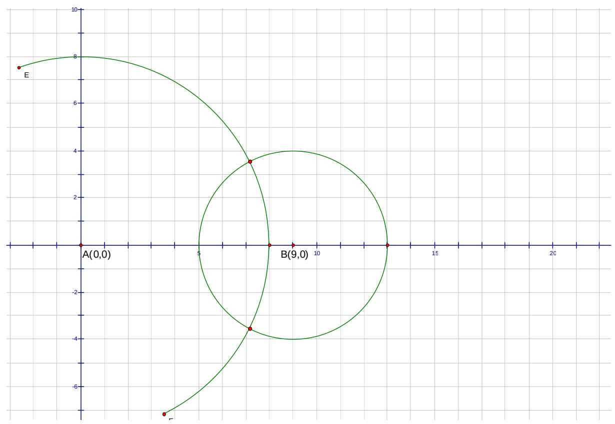


Locus Investigation Involving Distance

Materials needed are cm graph paper, calculator, compass, and ruler.

1. The challenge is to find any points P (x,y) that have the following distance property. The distance from P to A (0,0) is twice the distance from P to B (9,0). The technique we are going to employ to try and find these points is to use the compass to draw circles centered at A and B with the radius of the circle centered at A being twice the radius of the circle centered at B.

2. Using a piece of cm graph paper in the horizontal position, put the x-axis in the middle of the paper, and the y-axis just a few grid lines to the right of the left edge. Plot the points A (0,0) and B (9,0). As a first example, use the compass to lightly draw a circle centered at B with radius 4, and then a circle centered at A with radius 8. Your paper should look like the diagram at right. Notice it is not necessary to draw the entire circle in order to find the points of intersection, but appropriate arcs of the circles will suffice.



3. The points of intersection of the two circles clearly have the property of the points we are interested in. Mark those intersection points clearly, and then repeat the process by choosing different radii each time. Of course, the circle centered at A must always have a radius twice the radius of the circle centered at B. Note, if you choose a radius that is too small or too big, the circles will not intersect. Each time you change radii, mark the intersection points. Our goal is to see if all these intersection points have a pattern or lie on some sort of recognizable curve.

4. After you have done several of these, answer the following questions.
   a) What is the radius of the smallest circle centered at B that produces an intersection?

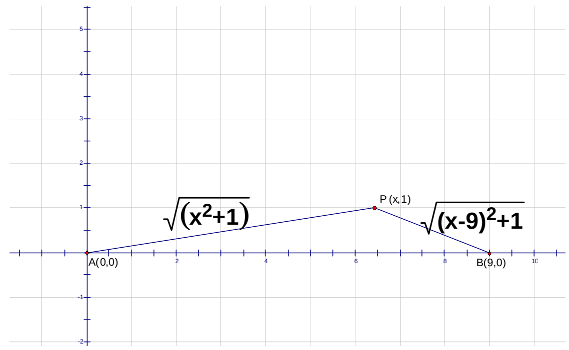
   b) What is the radius of the largest circle centered at B that produces an intersection?

   c) What points on the x-axis have the property that $2PB = PA$.

   d) Do you have a conjecture as to the locus of all the points?

5. Let's try and algebraically find the x-axis points by calling the unknown point $P = (x,0)$. The distance requirement $2PB = PA$ becomes the algebra equation $2\sqrt{(x-9)^2} = \sqrt{x^2}$. In the space below, square both sides, collect terms, and solve the resulting equation for x. Hopefully, your answers will agree with what you found when you did your compass work.

6. If you did step #5 correctly, your algebra should have yielded (6,0) and (18,0) as the x-intercepts. Let's repeat step #5 using another value for the y-coordinate of P other than zero. This time, let $P = (x,1)$, and thus verify that the distance condition yields the equation $2\sqrt{(x-9)^2+1} = \sqrt{x^2+1}$. Once again, in the space below, square both sides, collect terms, and solve the resulting quadratic equation. This time the solutions will not be rational, so 2-decimal place accurate answers will suffice. Plot the points you found on your graph, and see if they fit in with the pattern that the compass construction was exhibiting.



7. Since we know the x-values go from 6 to 18 we can also make the unknown value the x-coordinate and solve the distance relationship for the y-value. See the instructor to be assigned one of the x-values in the table below. Calculate the y-coordinate, and enter your result in the table and on the master table on the blackboard. Enter the data into the calculator and make a scatter plot. What do you think the locus looks like now?

x	7	8	9	10	11	12	13	14	15	16	17
y											

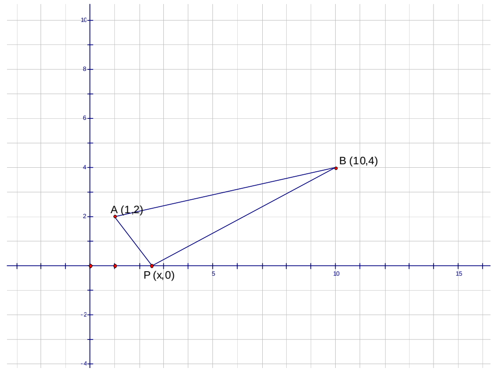
8. It is now time to find the equation for the locus of points with the distance condition $2PB = PA$. All we need to do is to follow the exact same steps as we did in tasks #5 and #6, but this time use the point $P = (x,y)$ rather than $(x,0)$ or $(x,1)$. Write the algebraic equation that equates with $2PB = PA$ in the space below, square both sides, and then solve for y. Because there is a y^2 , you will get two different expressions for y. Using the calculator, enter one expression into y_1 and the other into y_2. Choose an appropriate window to see the curve in its entirety, and then graph. Is this the curve you expected?

9. Advanced: Find the locus of points that are three times further from (0,0) as they are from (4,0).

Right Angle Investigation

Materials needed are cm graph paper, calculator, compass, and ruler.

1. Given line segment *AB* where *A* = (1,2) and *B* = (10,4) find any points *P* such that angle *APB* is a right angle. Let's start by trying to find any points on the *x*-axis that make angle *APB* equal to 90 degrees, and then we can expand from there. Start by plotting the points *A* and *B* carefully on a piece of cm graph paper. Also, put an arbitrary point *P* somewhere on the *x*-axis that looks like it might form a right angle. The coordinates for this point *P* are (*x*,0).



2. There are a few ways to find the coordinates of point *P*, with one of them being slope. Recall that lines are perpendicular to each other if and only if the product of their slopes is −1. The expression for the slope of *AP* can be written as $\frac{2}{1-x}$. In the space below, write down an expression for the slope of *BP*, and set up an equation that says that the product of the two slopes equals −1.

3. Hopefully, you have written an equation equivalent to $\left(\frac{2}{1-x}\right)\left(\frac{4}{10-x}\right)=-1$. This is not a trivial equation to solve, so you will have to take your time and do the algebra correctly. After doing the multiplication, the clearing of fractions, and the collecting of terms, you should notice that your equation is QUADRATIC. This means that you want to get the equation in the form $ax^2+bx+c=0$ in order to solve it by either factoring or the quadratic formula. In the space below, show all your work, and solve the equation above.

4. If you did the algebra above correctly, it should have resulted in the equation $x^2-11x+18=0$ and then the answers $x = 2$ and $x = 9$. Thus, *P* has two different possible locations on the *x*-axis at (2,0) and (9,0). Verify, that these two points do indeed produce a right angle at *APB*, and plot them on your graph.

5. In order to find more possible locations for point P that makes APB a right angle, we will look for points on other horizontal lines other than the x-axis. If the horizontal line $y = k$ is used, then the only change in the process used in steps #1– #4 will be to give point P the coordinates (x, k) instead of $(x,0)$. Go to the instructor and have one of the values for k in the chart below assigned to your group. Repeat steps #1– #4 and put the appropriate information into the table below and on the chart on the blackboard. Note: If the quadratic equation does not have rational solutions, round off answers to 2-decimal place accuracy.

Horizontal Line $y = k$	Point P	Slope AP	Slope BP	Equation	Solutions
$y = 0$	$(x,0)$	$\frac{2}{1-x}$	$\frac{4}{10-x}$	$\left(\frac{2}{1-x}\right)\left(\frac{4}{10-x}\right) = -1$	(2,0) and (9,0)
$y = -1$	$(x,-1)$				
$y = 1$	$(x,1)$				
$y = 3$	$(x,3)$				
$y = 5$	$(x,5)$				
$y = 6$	$(x,6)$				
$y = 7$	$(x,7)$				

6. Copy the solutions from the blackboard, and plot them carefully on your graph. You may also find another two points that correspond with $y = 2$ and $y = 4$ from just observation. Look to see if there appears to be any kind of pattern to these points and make note of it in the space below.

7. To generalize our work, all we need to use the point (x,y) for P and follow the exact same procedure. The only difference will be that instead of solving a quadratic equation for x that results from the slope relationship, you will need to put the resulting quadratic relation into the form $x^2 + bx + y^2 + dy + e = 0$. Show all your work below.

8. The final step is to use completing of the square for both the $x^2 + bx$ and the $y^2 + dy$ expressions in order to put this quadratic relation into the more recognizable form $(x-h)^2 + (y-k)^2 = r^2$. Do the work below, and identify in detail what the locus of points P actually forms.

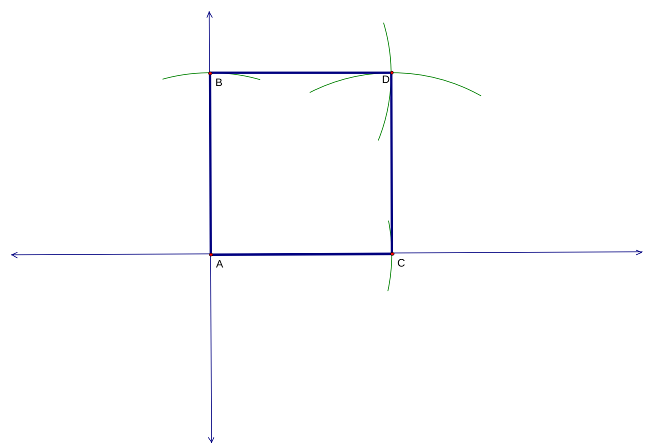
The Golden Rectangle

Preliminary Question: What number when diminished by its reciprocal is equal to one?

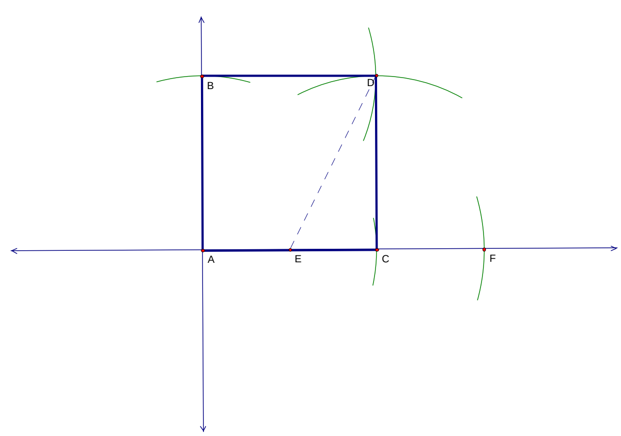
1. The Golden Ratio Phi is equal to $\frac{1+\sqrt{5}}{2}:1$, which is approximately 1.618:1, and is represented with the Greek letter ϕ. This ratio was well known to the Greeks who believed rectangles whose sides were in this ratio were the most pleasing to the eye. There are many places in nature and mathematics where this ratio comes up, and you are encouraged to read more on this topic. The Golden Rectangle is the only rectangle from which a square can be cut or added and the remaining rectangle will always be similar to the original.

2. The first part of this construction is to construct a square. Start by drawing a reasonably horizontal line in the middle of your paper and plot a point A on this line a few inches to the left of center. Construct a line perpendicular to the first line and passing through point A. (see worksheet on compass and straightedge constructions)

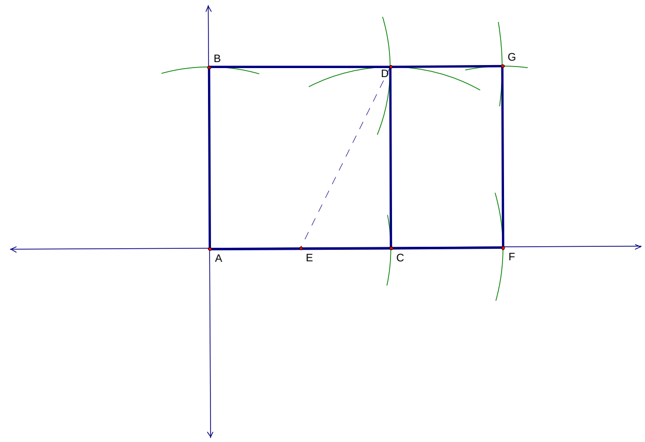
3. Construct arcs of a circle centered at A with a radius of approximately 3 inches that intersect both lines. Label the intersection point of the arc with the perpendicular line point *B*, and the intersection of the arc with the horizontal line point *C*. Now construct arcs of circles centered at *B* and C with exactly the same radius used for the previous arcs. These arcs intersect at point *D*, which is the 4th vertex of square *ABDC*. Connect the vertices, and your diagram should look like the one at right. This square will be used to construct a Golden Rectangle whose width is *AB*.



4. Bisect *AC* to find its midpoint *E*. (see worksheet on compass and straightedge constructions). Label the distance from *E* to *C* as *x* units and find the lengths of *CD* and ED in terms of *x*. Draw an arc of a circle centered at *E*, with radius *ED*, and passing through line *AC* at point *F*. Then determine the distance in terms of *x* of *AF*, and determine the ratio of *AF: AB*. Your diagram should look like the one at right.

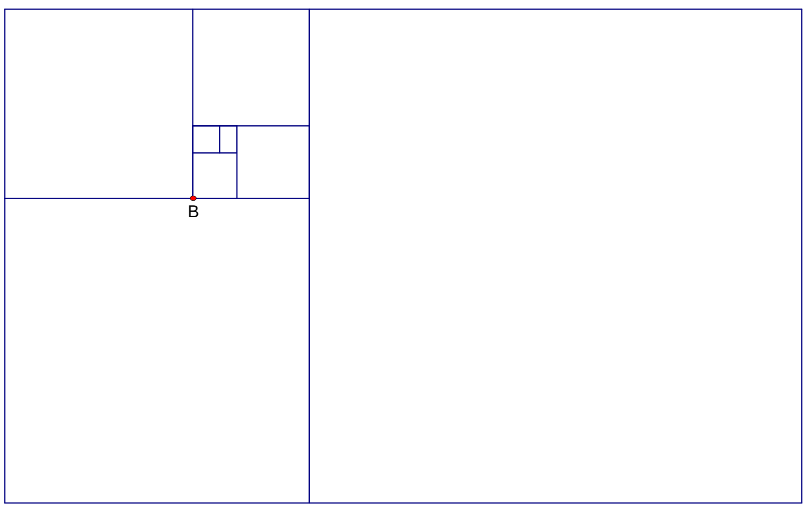


5. Hopefully your calculations revealed that $CD = 2x$, $ED = \sqrt{5}\,x$, $AF = x + \sqrt{5}\,x$, and that the ratio $AF{:}AB = \dfrac{x+\sqrt{5}\,x}{2x} = \dfrac{1+\sqrt{5}}{2} = \phi$. This is of course the ratio we needed for a Golden Rectangle, so all that is left to construct is the last vertex of the rectangle. This should be an easy construction that will be left to you to complete without any directions. Once the last vertex is constructed, connect the vertices to form a Golden Rectangle as shown to the right.



6. Prove that rectangle *CFGD* is a Golden Rectangle.

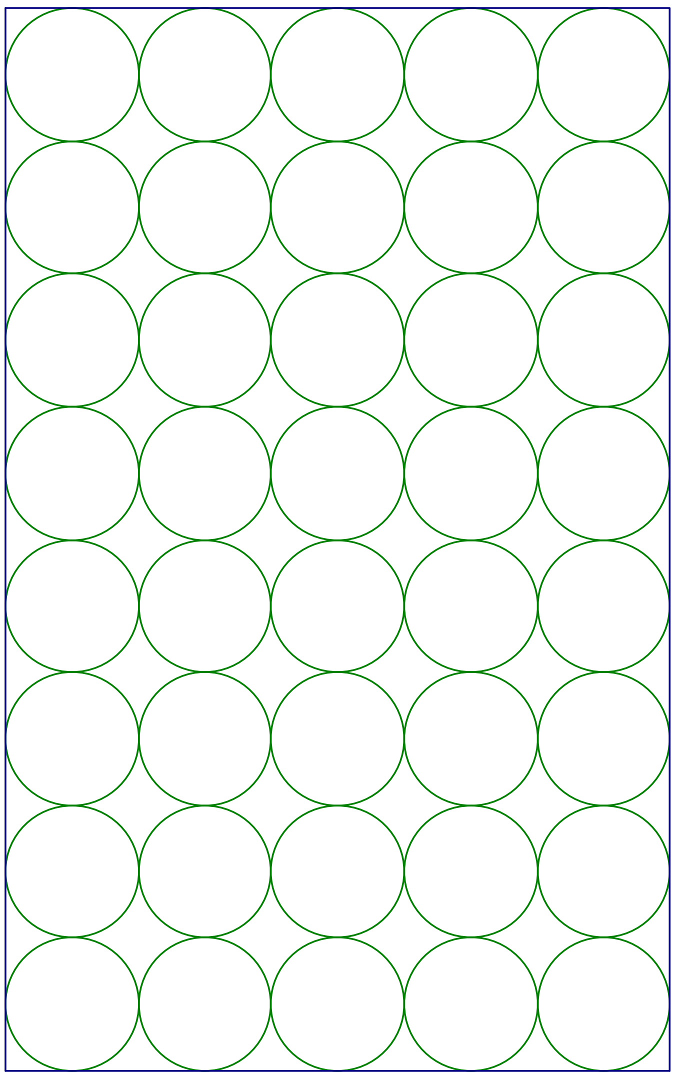
7. Actually, there are two Golden Rectangles in the diagram. The large rectangle AFGB and rectangle FGDC are both golden. The **challenge** for students with some extra time is to construct two more golden rectangles in this diagram by constructing squares in the appropriate location. Good luck. This process of adding and subtracting squares could be repeated indefinitely. Below is an example of a diagram with **<u>eight</u>** golden rectangles.



The Penny Packing Dilemma

Materials needed: Approximately 50 pennies or other congruent circles, paper, straight edge, calculator

The job at hand is to pack as many pennies into the rectangular space that has dimensions 5 penny diameters by 8 penny diameters. (Any circle size will do as long as dimension of rectangle is 5 diameters by 8 diameters). The obvious solution has a total of 40 pennies in the space with the pennies in the tangent position as shown. Is this indeed the most number of pennies that can fit into this space, or can you find a way to do even better? You must use a rectangle that is exactly 5 diameters x 8 diameters, so your first task is to construct a rectangle with those dimensions. Now see if you can fit more than 40 pennies into that space. (No stacking of pennies allowed. Just one layer) If so, give a clear mathematical argument how you know this can be done.



Bonus: What is the most pennies that can be packed into a 8 diameter by 16 diameter rectangular space?

An Interesting Quadrilateral

Materials needed are graph paper, protractor, compass, ruler, and scissors.

1. Choose a good scale for your graph paper so that the diagram is large after carefully plotting the points $P = (-15, 0)$, $Q = (5, 0)$, $R = (8, 21)$, and $S = (0,15)$.

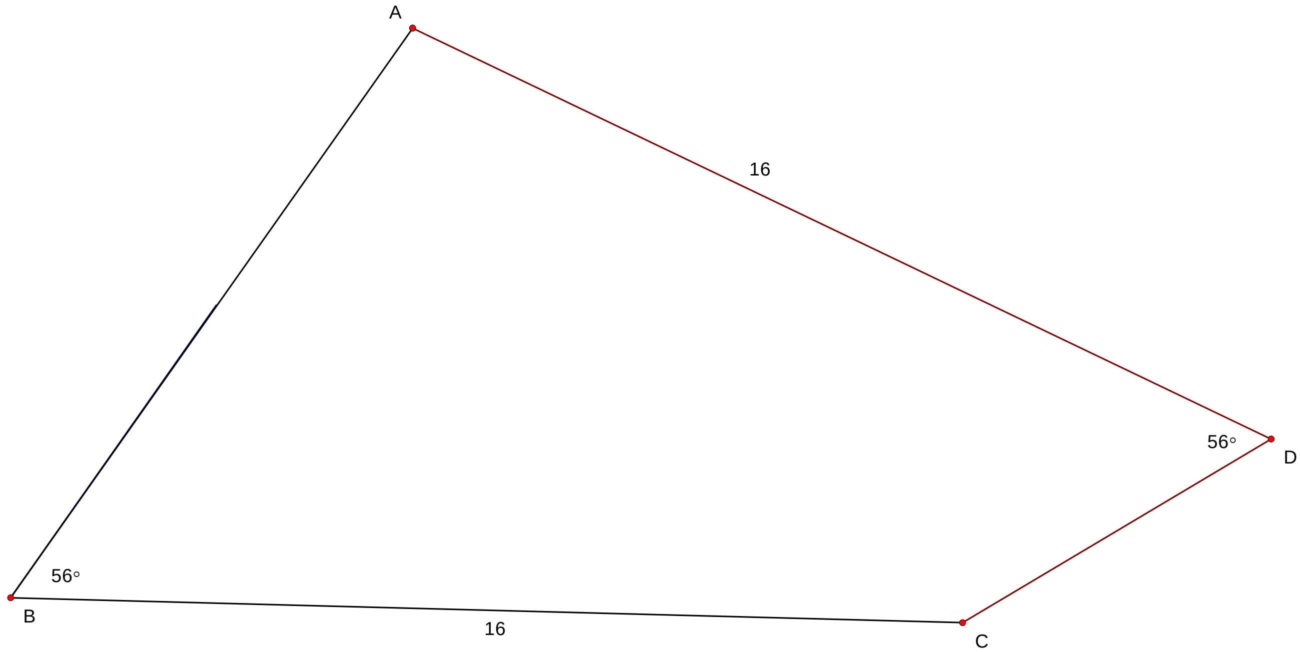
2. Measure the sides and the angles in this quadrilateral and note anything below that seems possibly interesting and worth investigating further.

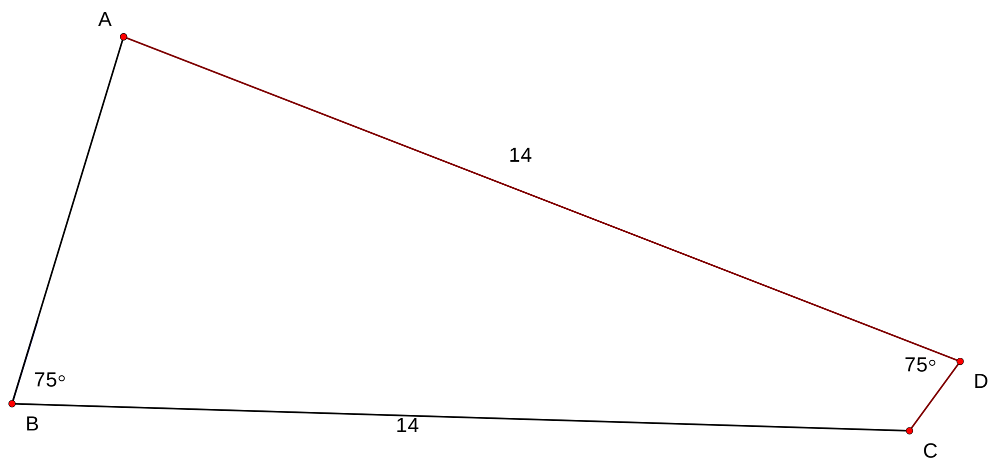
3. Hopefully you noticed that two opposite sides seem to be the same length, and also two opposite angles seem to have the same measure. Use the distance formula to confirm the distance equalities, and also try and confirm the angle equalities through some mathematical argument. Show work below.

4. What is interesting about this quadrilateral is that although two opposite angles are equal and two opposite sides are equal, the quadrilateral is clearly NOT a parallelogram. Explain why this does not contradict one of the sufficient and necessary conditions we have studied about parallelograms.

5. Construct diagonal *SQ* and look carefully at the triangles *RSQ* and *SPQ*. What do you notice about their sides and angles?

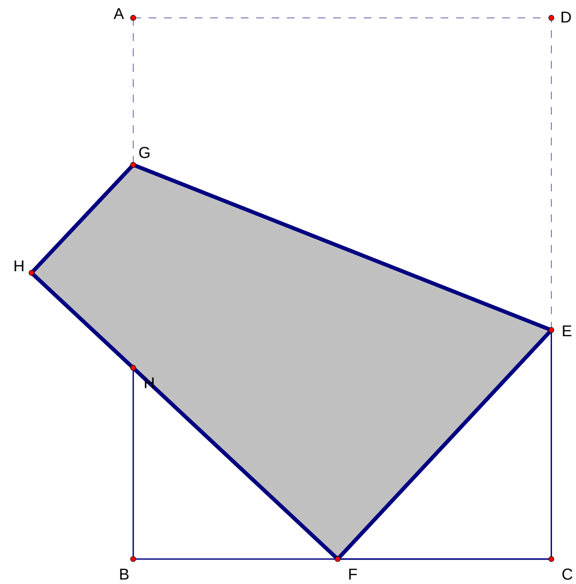
6. On the next page are two more examples of quadrilaterals that have one pair of opposite sides equal and a pair of opposite angles equal. Try and put together one of your own. Cutting and pasting is allowed, but there should be more than measuring evidence that the appropriate sides and angles are congruent.


A
16
56°
D
56°
B
16
C


A
14
75°
D
75°
B
14
C

A Paper Folding Investigation

Materials needed: 18 x 24 cm graph paper, ruler



1. Cut out the edges of the graph paper so that only the 18 x 20 cm graph paper is left. Hold the paper so that the 20 cm sides are vertical, and label the corners *A, B, C*, and *D* as shown in the picture at right. The corner at *D* is going to be folded so that it matches with a point *F* on edge *BC*. Let *DE* = 12 for this first example.

2. Since *DE* = 12, *EC's* length must be 8. Because this was a folding operation, what other length in the diagram must also equal 12?

   Using *EC* = 8 and *EF* = 12, determine the length of *CF* using the Pythagorean Theorem. Use you ruler to measure as a check on your calculated length.

3. Determine the area of triangle *EFC* and enter it into the chart below. Then do the necessary calculations to complete the rest of the chart.

Length of *DE*	11	12	13	14	15	16	17	18
Length of *EC*		8						
Length of *CF*								
Area of triangle *EFC*								

4. Enter the data from the chart on the previous page into your calculator and do a scatter plot of length *DE* versus the area of triangle *EFC*. Based on the data you have collected, how accurately can you declare the length of *DE* that maximizes the area of triangle *EFC*?

5. To find a more exact answer to the maximizing question above, let the length *DE* equal the variable *x*, and then write a function of *x* for the area of triangle *EFC*. Write your function below.

6. Enter your function into the calculator, graph the function for an appropriate domain, and then find the maximum of the function. Write your answer below.

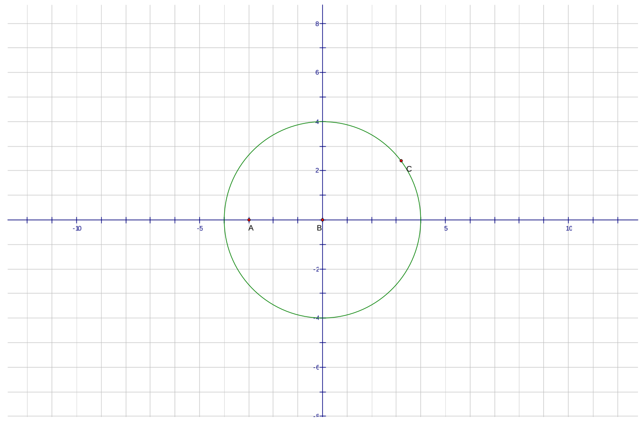
7. Do you notice anything interesting about this maximum area triangle?

8. As a follow-up, find the length of *DE* that will maximize the sum of the two triangular areas *EFC* and *BFH*. First prove that triangle *BFH* is similar to triangle *EFC*, and then use that fact to help find the total area.

SSSS Quadrilateral Investigation

Materials needed: cm graph paper, protractor, compass, straight edge

Question: A quadrilateral *ABCD* has side lengths of 3, 4, 5, and 6 cm. Find the quadrilateral that has the largest area.



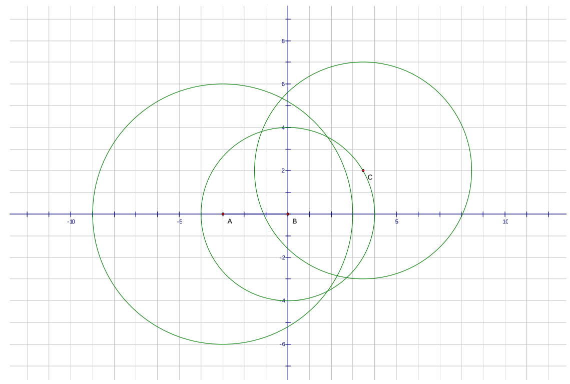
1. Turning you cm graph paper horizontally, locate the origin toward about two-thirds of the way down the center of the paper. Plot the point (−3, 0) and label it point *A*. Label the origin point *B*. Now construct a circle centered at *B* (0, 0) with a radius of 4 cm. Your paper should look like the one at right.

2. When you have reached this point, go to the instructor and get assigned one of the angles in the table below. Point *C* is going to be on the circle just constructed and in the first quadrant. The angle you are assigned will be the angle formed by *BC* and the positive *x*-axis.

Angle in degrees of *BC* and positive *x*-axis	10	20	30	40	50	60	70	80	90
Estimated coordinates of point *C*									
Actual coordinates of point *C*									
Measured length of *AC*									
Calculated length of *AC*									

3. Use the protractor to draw a ray from the origin that makes the angle assigned to you from the table above. The intersection of your ray and the circle centered at the origin is point *C*. Estimate the coordinates of point *C* and enter them into table above. Also, use trig to calculate the coordinates of *C* accurate to 2-decimal places, and enter that into the table above. Use the ruler to measure the distance *BC*, and also use your calculated measurements and the distance formula to calculate the distance *BC* with 2-decimal place accuracy. Enter the results above.

4. Use a compass to draw a circle centered at *C* with a radius of 5 cm. Also use a compass to draw a circle centered at *A* with a radius of 6 cm. You picture should look like the one at right.

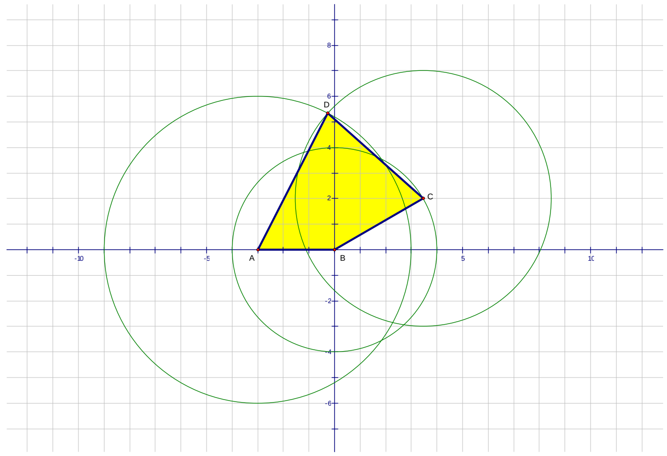


5. Hopefully, it should be clear to you why the 4th vertex *D* of this 3-4-5-6 quadrilateral must lie at an intersection point of the last two circles you constructed. Label the intersection point above the *x*-axis as point *D*, and connect the points *ABCD* to form the quadrilateral.

6. Your job now is to find the measure of the four interior angles in your quadrilateral and to determine its area. Use a protractor to measure the interior angles. Draw diagonal *AC* to split the quadrilateral into two triangles, and use a ruler to obtain as accurate an estimate as possible for the areas of the triangles and thus the area of the quadrilateral. Record your results in the appropriate column in table below, and also on the master table for the class.

Angle in degrees of *BC* and positive *x*-axis	10	20	30	40	50	60	70	80	90
Angle *ABC*									
Angle *BCD*									
Angle *CDA*									
Angle *DAB*									
Estimate for the area of quadrilateral *ABCD*									

7. Once the master table has been completed, copy the data into your own table. Enter the angle and area data into the calculator as *L1* and *L2*, and do a scatter plot of that data. In the space below, write down what you can ascertain from the data as far as determining a maximum area. A picture of one of the many 3-4-5-6 quadrilaterals is shown at right.



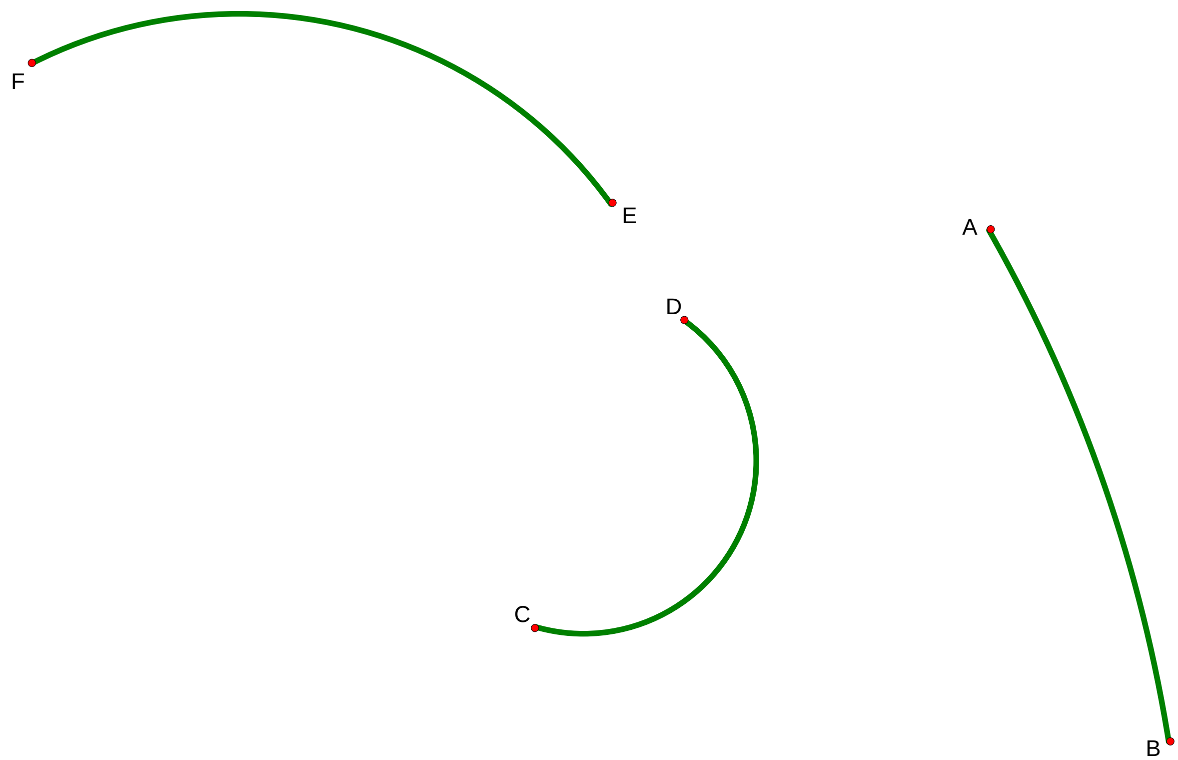
8. Repeat step #3 for an angle of 65 degrees. This is approximately the angle that produces the maximum area quadrilateral. Do you notice anything interesting about the four interior angles?

9. **For advanced students**: Instead of using a specific angle as we have done up to this point, call the angle *x* degrees, and find an area function for the quadrilateral as a function of *x*. Enter this function into the calculator and find its maximum. Is your answer consistent with the data from this lab? The Law of Cosines is involved in writing this function.

Finding the Center and Radius of Circles from Arcs

Materials needed: Pencil, compass and ruler.

1. The challenge in this worksheet is to find the center of a circle when only given an arc of that circle. Your tools are to be a compass and a straight edge only, and you are to use any constructions you feel appropriate to get as accurate an answer as possible. There are three such arcs below. You are to make a copy of these arcs on another piece of paper before doing your constructions to find the center. Bear in mind, that it is possible that a radius of one or more of these circles is larger than the width of one piece of paper.



2. Once you have located the center for each of the circles above, use the ruler or a yard stick to measure the radius as accurately as possible in cm. Record answers below.

3. Try and accomplish the same task as above with the same three arcs, but this time, do not use any compass or straight edge constructions. Once again, first make a copy of the arc on another piece of paper. See how your answers compare.

Fermat Point Investigation

Materials needed are cm graph paper, compass, protractor, calculator, and ruler.

Problem - Given the triangle *A*(0,0), *B*(2,8), *C*(10,0), find the point *P* such that the sum of the distances *PA* + *PD* + *PC* is a minimum.

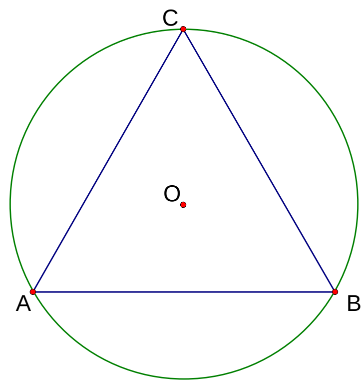
1. On cm graph paper, construct the triangle defined above by plotting the three points. In looking for the minimum distance point, it is not unreasonable to suspect one of the special points we have examined already. Use algebra to find the **circumcenter O**, **centroid G**, **orthocenter H**, and **incenter I** for this triangle. You may need some review on how to find these points, so ask the instructor for help if necessary. Record the coordinates for each, plot them carefully on your sketch, and then **compute** the sum of the distances to the three vertices for each using the distance formula. Also, measure with a protractor the three central angles formed by **each** of these points. For instance, for the centroid, you will find angles *AGB, BGC, and CGA.*

Point	Coordinates of Point	Sum of distances to vertices	Measure of 3 central angles
Circumcenter			
Centroid			
Orthocenter			
Incenter			

2. You should notice that the sum of the distances to the vertices is smallest for the incenter. Although the incenter is NOT the minimum distance point we are looking for, it is closest of the four tested. What do you observe about the measure of the three angles for the incenter compared to the other three points?

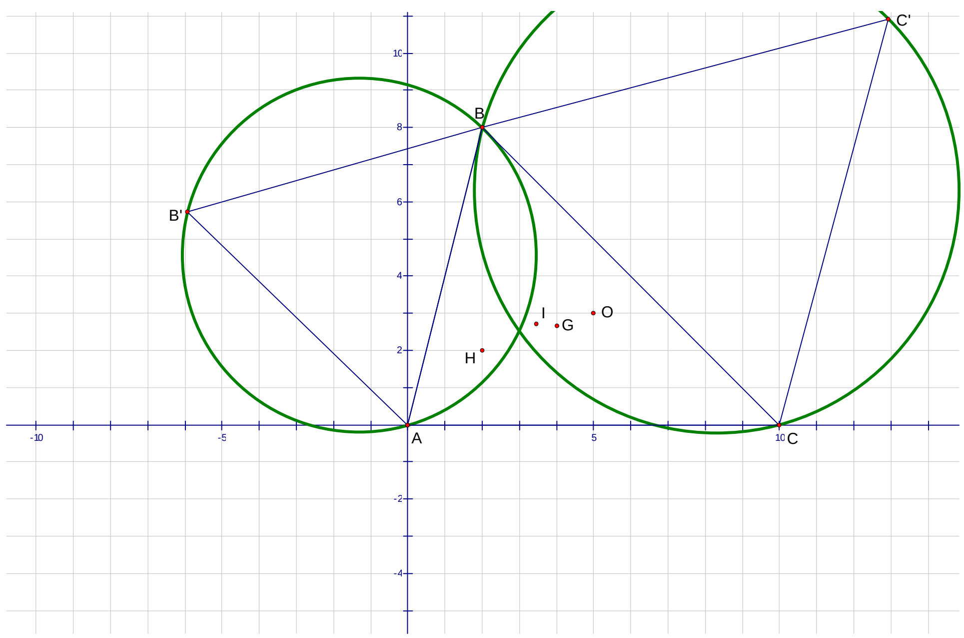
   If you noticed that the angles are closest to being all equal to 120 degrees, you are very observant. It turn out, that this is indeed the criteria we need to meet in order to find the minimum distance point, called the ***Fermat point,*** (or ***Steiner Point***). We need to find a point *F*, such that angles *AFB*, *AFC*, *BFC* all equal 120 degrees.

3. In order to determine the construction that will enable us to find the Fermat point, First, let's revisit an earlier problem. On a separate piece of paper, draw an equilateral triangle ABC reasonably accurately, but not taking the time to do a full blown construction. Find the ***circumcenter*** ***O*** of this triangle, and construct the ***circumcircle***. Your sketch should now look like the one to the right. This construction is the key to finding the Fermat point of a triangle. Put a random point P anywhere on the minor arc AB. Connect P to A and B, and measure the angle APB. Can you explain why any position of P on this arc yields a 120 degree angle?



Now measure the sum of the lengths $AP + BP$ and compare to length PC. Do this for a few different locations of P on the arc. Interesting result isn't it? It turns out that $AP + BP = CP$. If you want to see a proof of this, go to the last page of this lab entitled Ptolemy's Theorem.

4. We are now ready to locate the Fermat Point in our original triangle. Go back to that sketch, and draw an equilateral using side AB as one of the sides, and making sure that the triangle is attached to the outside of ABC. Repeat this for side BC, so that there are now two equilateral triangles attached to the original. Locate the circumcenters for these two triangles and draw the circumcircles for each. Your drawing should look like the one at right. Notice that the circles intersect twice. Once at a vertex, but they also intersect inside triangle ABC. Call this point of intersection F, and measure angles FAB and FBC. Hopefully you are not surprised by the result because of our work above in step #3. This point is the **Fermat Point** we were looking for, with coordinates (2.986, 2.524) (rounded to 3-decimal places). The sum of the distances to the vertices for this point is approx 16.9282 cm. You can see why point I came the closest in this triangle by looking at the location of F.



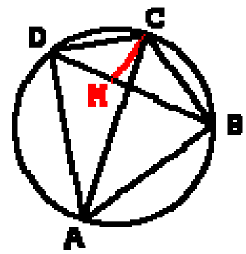
Ptolemy's Theorem

Let a quadrilateral *ABCD* be inscribed in a circle. Then the sum of the products of the two pairs of opposite sides equals the product of its two diagonals. In other words,

$$AD \cdot BC + AB \cdot CD = AC \cdot BD$$

This classical theorem has been proven many times over. Following is the simplest proof I am aware of.

Proof



On the diagonal *BD* locate a point *M* such that angles *ACB* and *MCD* are equal. Since angles *BAC* and *BDC* subtend the same arc, they are equal. Therefore, triangles *ABC* and *DMC* are similar. Thus we get $CD/MD = AC/AB$, or $AB \cdot CD = AC \cdot MD$.

Now, angles *BCM* and *ACD* are also equal; so triangles *BCM* and *ACD* are similar which leads to $BC/BM = AC/AD$, or $BC \cdot AD = AC \cdot BM$. Summing up the two identities we obtain

$$AB \cdot CD + BC \cdot AD = AC \cdot MD + AC \cdot BM = AC \cdot BD$$

Important result from Ptolemey's Theorem

> Let $A_1A_2A_3$ denote an equilateral triangle inscribed in a circle. For any point *P* on the circle, show that the two shorter segments among PA_1, PA_2, PA_3 add up to the third one.

Solution

Let *s* denote the length of the side of the given triangle. By Ptolemey's Theorem we have

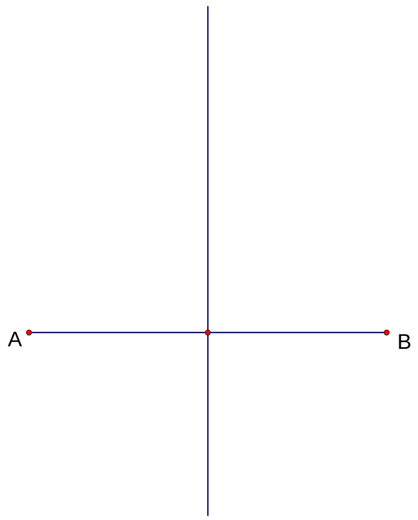
$$s \cdot PA_1 = s \cdot PA_2 + s \cdot PA_3$$

Therefore,

$$PA_1 = PA_2 + PA_3$$

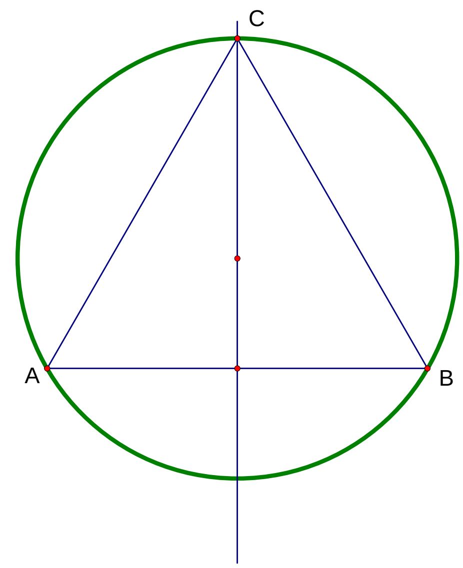
Constructing an Equilateral Triangle and its Circumcircle

Materials needed are compass, protractor, and a straight edge.



1. On a blank piece of paper, draw a line segment *AB* approximately 10 cm long in the middle of the page. Construct the perpendicular bisector of this segment using the compass and straight edge techniques learned earlier. Make sure the bisector extends above the line segment a length approximately equal to the length of the original segment. Your sketch should look like the one at right.

2. Based on your knowledge of the properties of an equilateral triangle and only using the compass and straight edge tools available, can you think of the next step? If so, execute it, and explain below in detail why your construction works to complete the equilateral triangle. If not, ask a neighbor or the instructor for a hint.



3. We now want to circumscribe the equilateral triangle *ABC* by finding the ***circumcenter***. What lines in a triangle are concurrent at the circumcenter?

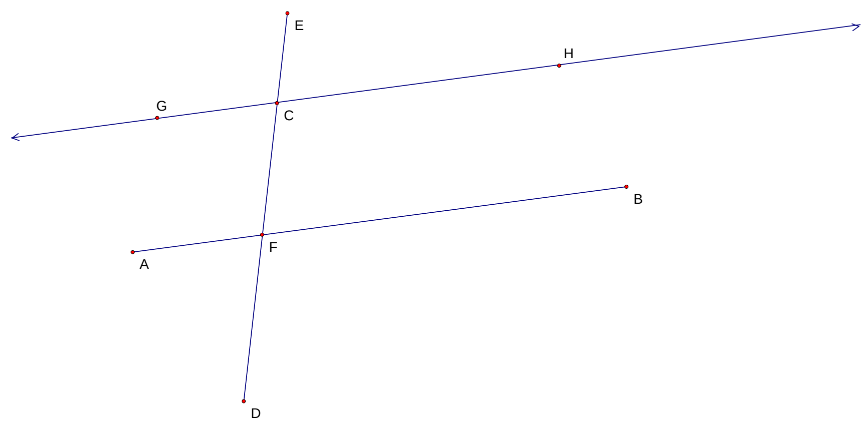
   Execute the necessary construction to find the circumcenter, and then use the compass to draw the circumcircle. Your sketch should look like the one at right without some of the construction lines.

4. Mark a random point *P* on the minor arc *AB*. Lightly draw the segments *PA* and *PB*, and then measure the angle *APB*. Compare your answer with all of your neighbor's, and try and write an explanation for what you have observed in the space below.

Parallel Lines

Materials needed are compass, protractor, and a straight edge. Preliminary work includes constructions on page 4 of this document.

1. Starting with a blank piece of paper, draw a line segment *AB* across the width of your paper at about the middle. Mark a point *C* a random distance above *AB*. Using the construction techniques you practiced on page four, construct a line parallel to *AB* that passes through point *C*. Label points *G* and *H* to the left and right respectively of point *C* and on the parallel line. As shown on page four, Point *F* is the intersection of the line used in the construction process and *AB*. This gives you two parallel lines on the page looking like the sketch at right, without the construction circles showing.



2. Using a protractor measure angles *GCE, HCE, GCF*, and *HCF*, and record results below.

   Also measure angles *AFC, BFC, AFD*, and *BFD* and record results.

3. Make observations about your results, record them below, and compare those observations with a neighboring group.

4. Hopefully you noticed several pairs of angles that were the same measure or certainly within the error that might occur when doing constructions by hand. These equal pairs of angles have been given specific names for the situation in which parallel lines are crossed by another line or what is more formally called a ***transversal***. The specific names for these pairs of equal angles are ***corresponding angles***, ***alternate interior angles***, and ***alternate exterior angles***. In the space below, write pairs of angles from those that you measured in step #2, that you feel belong to each category below. Remember to write them as a pair. An angle will belong to more than one pair and category.

   Corresponding Angles:

   Alternate Interior Angles:

   Alternate Exterior Angles:

5. Another category is called ***interior angles on the same side***.

   What do you think the word interior is referring to?

   What do think same side is referring to?

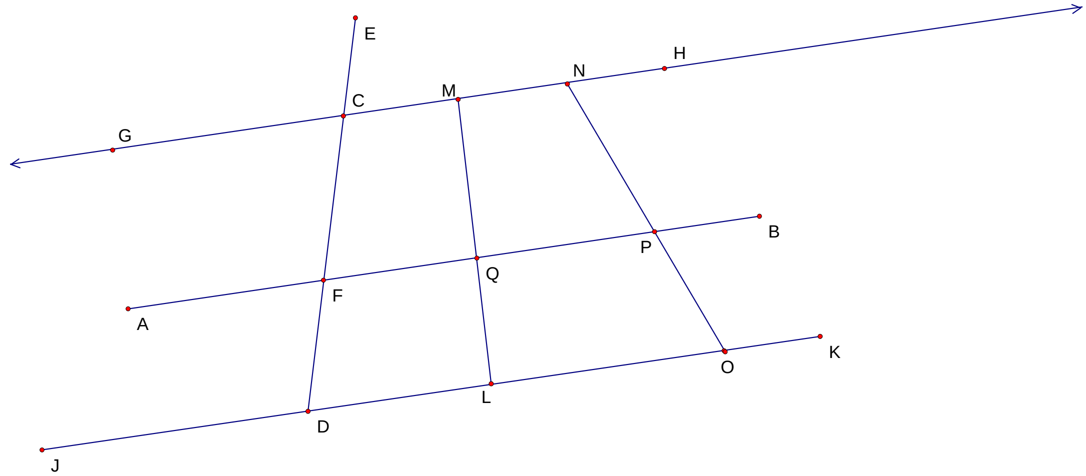
   What is true about pairs of angles that are interior and on the same side?

   List any pairs that fall into that category.

6. Assuming that angle ECH = 65 degrees, find the other seven angles that you measured in step #2.

7. For good practice, do problem 6 on page 32, and problem 3 on page 33, in the Math 2 materials.

8. Using your existing sketch, draw a third parallel line segment *JK* that passes through point *D*. Notice the use of the word draw instead of construct. In order to save some time and make the sketch less cluttery, a careful drawing of the third parallel line rather than a formal construction is requested here. Also, draw two more transversal segments *LM* and *NO* through the three parallel lines with points *L*,*M*,*N*,and *O* on the lines as shown in the diagram at right. . Label the intersection points of segments *LM* and *NO* with the third parallel line *P* and *Q*, as shown.



9. Measure in cm and record the lengths of the segments indicated in the table below.

CF	***FD***	***MQ***	***QL***	***NP***	***PO***

10. No obvious relationship seems to be emerging, but examine the ratios $\frac{CF}{FD}, \frac{MQ}{QL}, and\ \frac{NP}{PO}$ to help see what is going on.

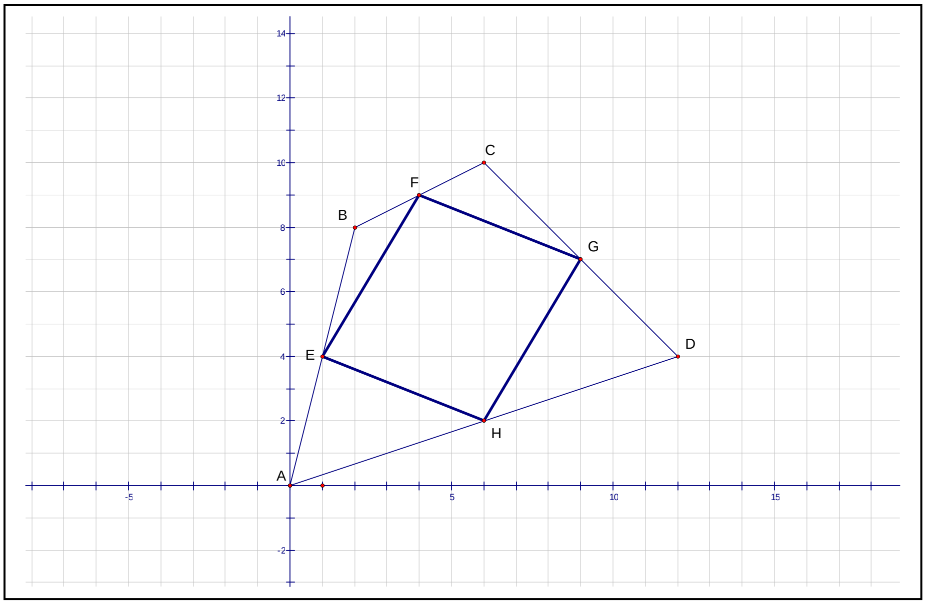
11. Try and finish the following statement: Given three parallel lines, the ratio of the lengths of the segments that are intercepted on one transversal are ________________________________

____. This statement is know as the ***Three-Parallels-Theorem.***

Midpoint Quadrilaterals

Materials needed: Graph paper, straight edge, and compass.

1. Start by plotting the four points $A = (0,0)$, $B = (2,8)$, $C = (6,10)$, and $D = (12,4)$ and forming the quadrilateral *ABCD*. Determine the coordinates of the four midpoints *E, F, G,* and *H* of the sides, plot those midpoints, and then connect them to form the ***midpoint quadrilateral*** *EFGH*. Your sketch should look like the one at right.



2. Does the midpoint quadrilateral seem to exhibit any special properties?

   Do any necessary calculations in the space below to verify or disprove any conjecture you made above. State any final conclusion you have arrived at about *EFGH*.

3. Now choose a random quadrilateral of your own with vertices anywhere you want. Once again, form the midpoint quadrilateral and look carefully for any special properties it has. Perform the necessary calculations to prove or disprove your conjecture, and write down the results below. When you are finished this step, go to the instructor and get assigned one of the quadrilaterals in the table in step #4

4. Once you have been assigned one of the quadrilaterals in the table below, your first challenge will be to determine coordinates of the vertices for a sample quadrilateral of the type you have been assigned. Some of the quadrilaterals are much easier to deal with than others.
Once you have coordinates for the vertices of the quadrilateral, find the midpoint quadrilateral and determine if it is of a special class. Show all your work. Once you have "typed" the midpoint quadrilateral, enter its type in the table below.

Parent Quadrilateral	Random	Square	Rhombus	Rectangle	Parallelogram	Isosceles Trapezoid	Kite
Midpoint Quadrilateral							

If this task goes quickly, then you can make it more of a challenge by trying to use literal constants as coordinates rather than specific numbers. This would lead to a coordinate proof rather than using a specific example to make a generalization. For instance, the general coordinates for a generic right triangle could be (0,0), (*a*,0), and (0,*b*). Although a vertex is located at the origin, and the perpendicular sides are parallel to the axis, there is no loss of generality in using these coordinates for a coordinate proof about right triangles. Why?

If this seems too difficult, then work out the details for another specific example of the quadrilateral you have been assigned. Once you are convinced of the type for the midpoint quadrilateral, enter the type in the table on the board.

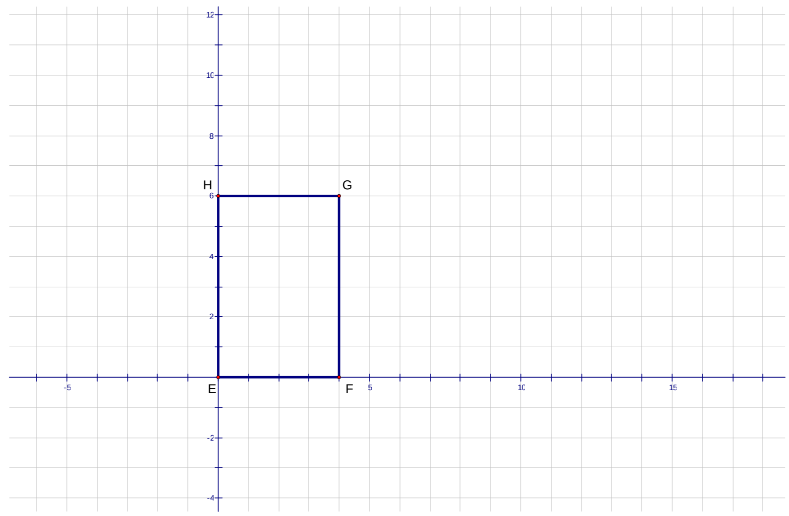
5. The next challenge is to understand why the parent quadrilaterals in the chart have specific type of midpoint quadrilaterals. It turns out there is a relatively easy explanation that has to do with a certain property of the parent quadrilateral. Perhaps you have an idea what this is, and can proceed to write an explanation in the space below. If not, look at the hint on the next page.

6. The hint is to draw in the diagonals of the parent quadrilateral and to think about the ***"midline theorem"*** for the triangles that are created. Hopefully, you can write an explanation in the space below explaining what is happening and why.

7. The following converse question about midpoint quadrilaterals is a very interesting one. What if quadrilateral $E = (0,0)$, $F = (4,0)$, $G = (4,6)$, and $H = (0,6)$ is given as the midpoint quadrilateral of a parent quadrilateral *ABCD*. Find coordinates for *ABCD*, and determine if the "parent" is unique. If not, how many different parents are there, and what do they have in common.

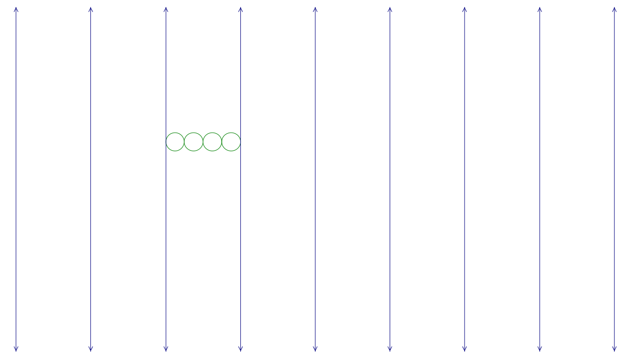
   Start by carefully plotting *EFGH*, and then think carefully about how you might construct the parent.

   Good Luck.



Tossing Pennies

Materials needed. Pennies, ruler, scotch tape



1. Using a blank piece of paper, make ruled lines spaced out 4 penny diameters apart. After making copies thru tracing, scotch tape several pieces of ruled paper together in order to form a surface large enough for the experiment of tossing a penny onto the surface. When finished, your surface should look like the one at right.

2. The experiment consists of tossing pennies onto the surface to see if it crosses any of the lines. You and your partner should repeat this experiment at least 25 times and record how many times the penny lands in open space and how many times it touches a line. Putting your surface up against a wall helps keep the pennies on the surface, and also you may want to toss more than one at a time. Record these numbers in the table below, and also on the board.

Group Number	1	2	3	4	5	6	7
Tosses with no lines touched							
Tosses with a line touched							

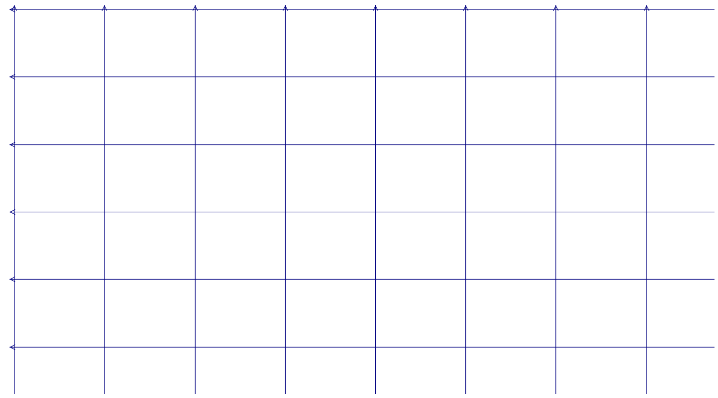
3. Total the number of attempts and the number of successes (not hitting a line), and use these numbers to **estimate** the theoretical probability of a penny not hitting a line.

4. Determine the theoretical probability using an area argument as to where the penny could land in order to not hit any of the lines. How does your theoretical probability compare to the estimate from the class experiment?

   How could we have done a better job modeling this problem?

   Is there a way to model this problem using the calculator and its random number generating ability? Explain

5. Draw horizontal lines three penny diameters apart so that your surface looks like the one at right. Once again, you and your partner should take turns tossing pennies onto this surface to determine how often it lands in open space without touching a line. Do this at least 25 times and record your results below and on the board.



Group Number	1	2	3	4	5	6	7
Tosses with no lines touched							
Tosses with a line touched							

6. Total the number of attempts and the number of successes (not hitting a line), and use these numbers to **estimate** the theoretical probability of a penny not hitting a line.

7. Explain why the coordinate pair (4*rand, 3*rand) generated from the calculator would be useful in a calculator simulation for modeling this problem.

   Work with your partner to generate at least 25 coordinate pairs (4*rand, 3*rand), determine which ones represent landing in open space versus hitting a line. Add these numbers to the totals in the table above and observe whether they are consistent with the coin tossing estimates.

8. Determine the theoretical probability using an area argument as to where the penny could land in order to not hit any of the lines. How does your theoretical probability compare to the estimate?